Anisotropic Fast-Marching methods

With applications to curvature penalization

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February 2, 2017

Mathematical Coffees, Huawei-FSMP

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This work was partly funded by ANR JCJC NS-LBR

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Example : Dijkstra's algorithm

For each $p \in X$ let $\mathrm{Neigh}(p) \subseteq X$ be a collection of neighbors, and $\delta(p, q)$ the corresponding edge lengths.

$$\Lambda U(p) := \min_{q \in \mathrm{Neigh(p)}} U(q) + \delta(q, p).$$

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Example : upwind discretization of $\|\nabla u\|^2 = s^2$

Assume that $X \subseteq h\mathbb{Z}^d$ is a cartesian grid, and let (e_i) be the canonical basis. Define for $U \in \mathbb{R}^X$, $p \in X$

$$HU(p) := h^{-2} \sum_{1 \leq i \leq d} \max\{0, U(p) - U(p + he_i), U(p) - U(p - he_i)\}^2.$$

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Setting: Finsler geometry

Consider a domain, a metric, and a speed function

$$\Omega \subseteq \mathbb{R}^d, \qquad \mathcal{F} : \overline{\Omega} \times \mathbb{R}^d \to [0, +\infty], \qquad s : \overline{\Omega} \to]0, \infty[.$$

Define for each smooth path $\gamma : [0, 1] \to \overline{\Omega}$

$$\mathrm{length}_{\mathcal{F}}(\gamma) := \int_0^1 \mathcal{F}_{\gamma(t)}(\dot\gamma(t)) \frac{\mathrm{d}t}{s(\gamma(t))}.$$

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Objective: compute a front arrival time

Given a set of seeds $S \subseteq \overline{\Omega}$ compute $u : \overline{\Omega} \to \mathbb{R}$ defined by

$$u(p) := \inf\{\text{length}_{\mathcal{F}}(\gamma);\ \gamma(0) \in S, \gamma(1) = p\},$$

and extract the corresponding minimal paths.

Using notations Ω (domain), S (seeds), u (front arrival time), $\mathcal{F}$ (metric), s (speed function).

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Bellman's optimality principle

$$q \in V \subseteq \Omega \setminus S \quad \Rightarrow \quad u(q) = \inf_{p \in \partial V} u(p) + d_{\mathcal{F}}(p, q).$$

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Discretization

Let $X \subseteq \Omega$ and $\partial X \subseteq \mathbb{R}^d \setminus \Omega$ be finite sets. Let $V(p)$ be a polytope enclosing each $p \in X$, with vertices in $X \cup \partial X$. Define

$$\Lambda U(x) = \min_{q \in \partial V(p)} \mathcal{F}_p(q - p) + \mathrm{I}_{V(p)}\, U(q),$$

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- Causality is equivalent to the <u>acuteness</u> of $V(p)$ w.r.t. $\mathcal{F}_p$.

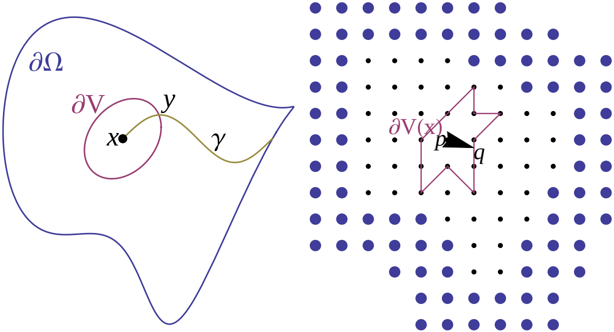

Figure: Illustration of Bellman's optimality principle, and of its discretization.

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- Has few vertices. ($\Rightarrow$ complexity)
- Has small vertices. ($\Rightarrow$ accuracy)

Voronoi-diagrams for 3D Riemannian metrics

Needle-like

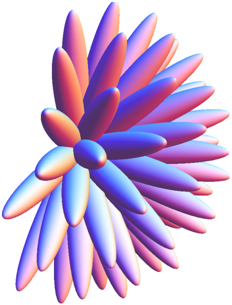
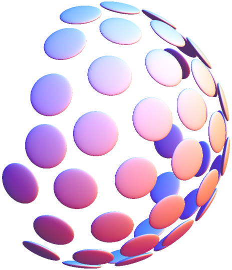


Plate-like



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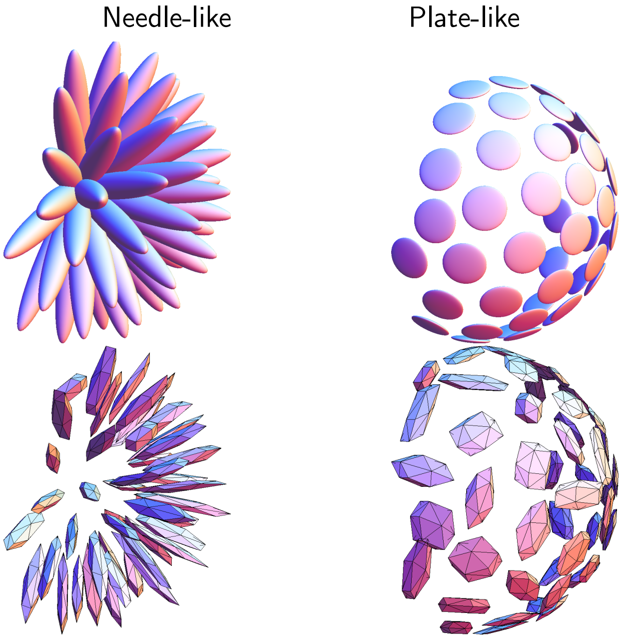


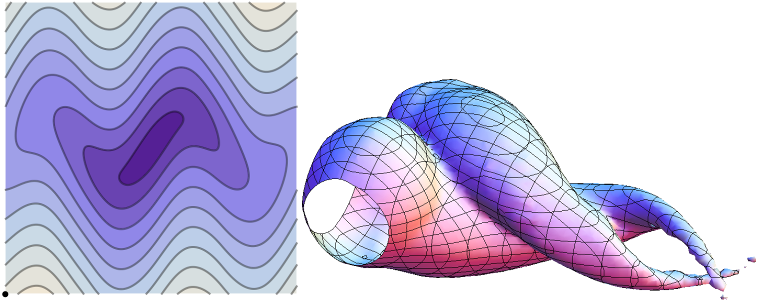


Figure: Some level sets of 2D and 3D riemannian distance maps.

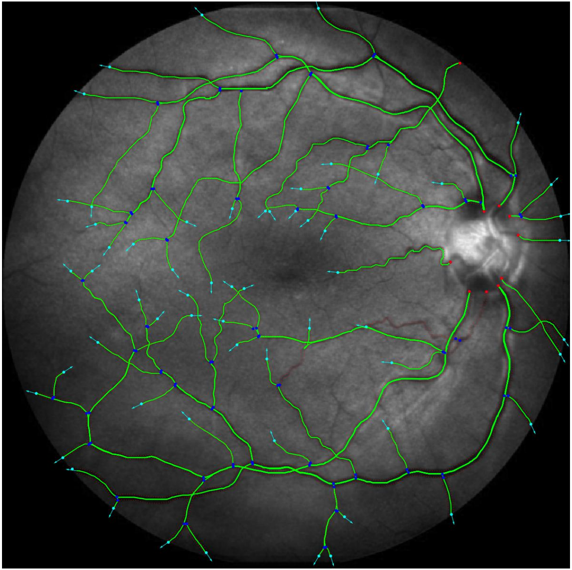


Figure: Segmentation of retina vessels. G. Sanguinetti, E. Bekkers, R. Duits, M.H.J. Janssen, A. Mashtakov, J.M. Mirebeau, Sub-Riemannian *Fast Marching in SE*(2), CIARP 2015.

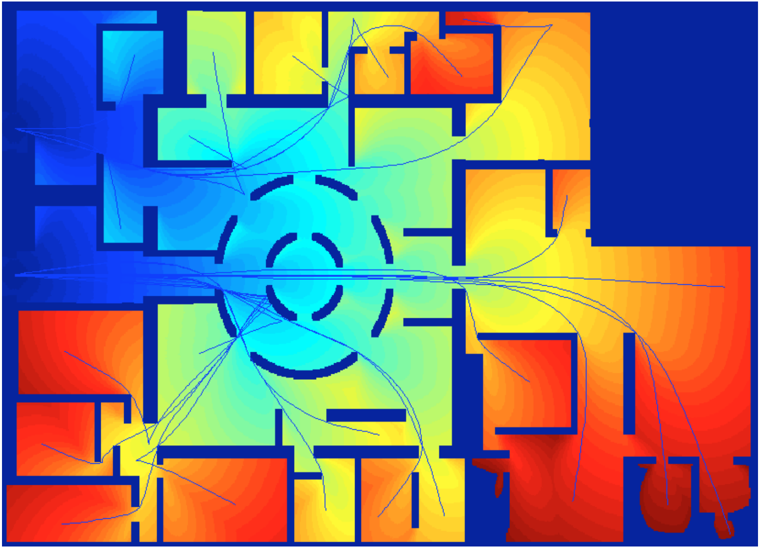
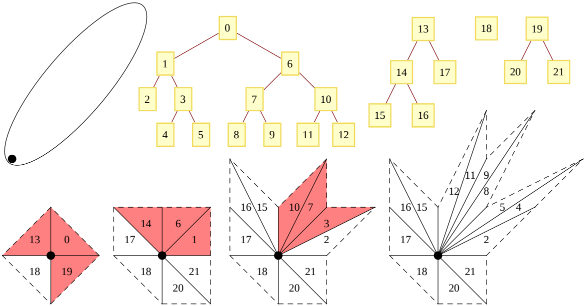


Figure: Shortest way out of centre Pompidou, using a Reeds-Shepp sub-riemannian metric. Note the many cusps.

Stencil refinement strategy for 2D Finsler metrics



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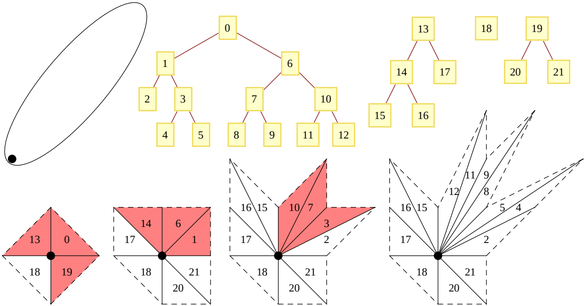

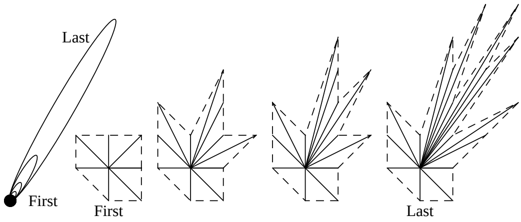

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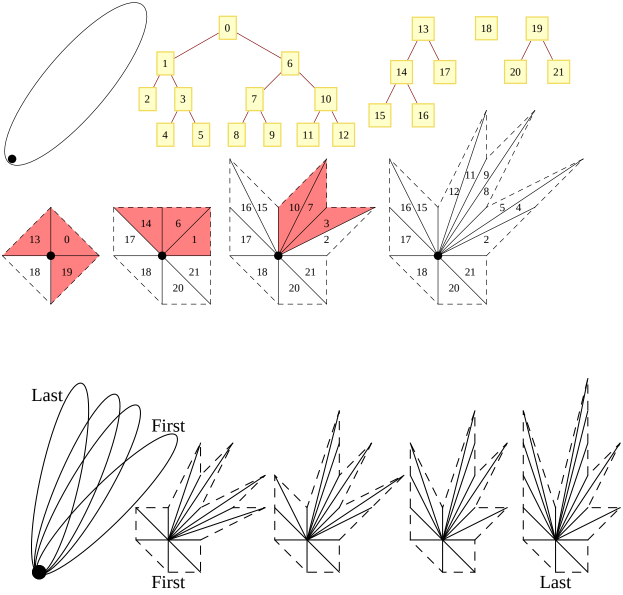

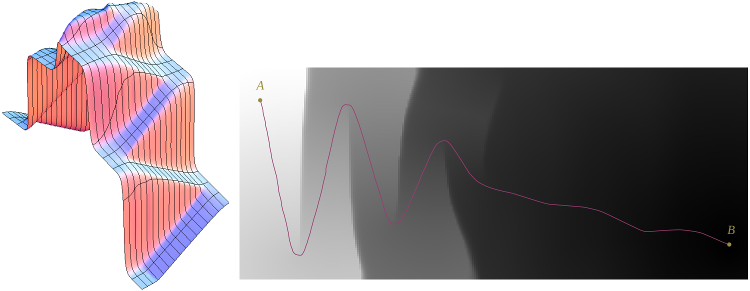

Figure: Finsler metrics can encode asymmetrical situations, e.g. ascent is harder than descent

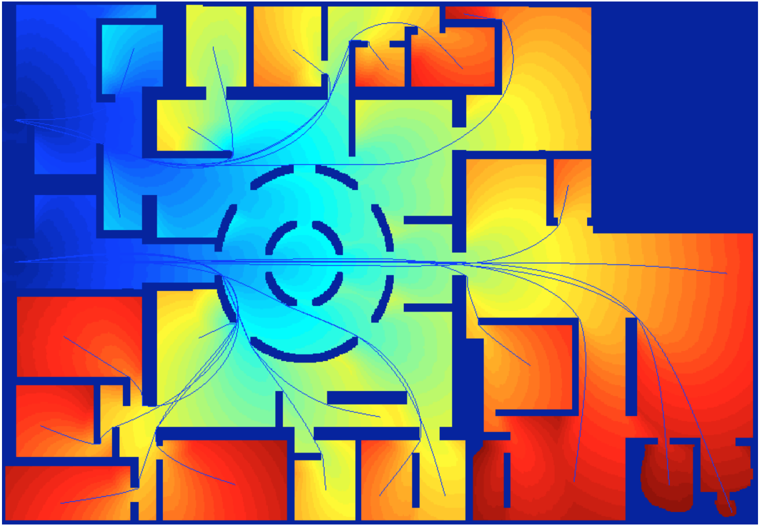

Figure: Shortest way out of centre Pompidou, using a Reeds-Shepp sub-riemannian metric modified to remove the reverse gear.

Conclusion on Semi-Lagrangian

Pros:

- Geometrical interpretation.
- Stencil recipes for 2D Finsler or 3D riemannian metrics on grids.

Cons:

- No good stencil recipe for 3D Finsler metrics, or for unstructuted meshes.
- A bit costly (iterate over all facets of $V(p)$ of all dims).
- Rather complex implementation in dimension ≥ 3.

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Generalized eikonal equation

Front arrival times are the unique viscosity solution to

$$\mathcal{H}_p(\nabla u(p)) = s(p)^2,$$

with $u = 0$ on S and outflow conditions on $\partial\Omega$.

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Discrete point set: a grid of scale $h > 0$

$$X := \Omega \cap h\mathbb{Z}^d, \qquad \partial X := (\mathbb{R}^d \setminus \Omega) \cap h\mathbb{Z}^d.$$

Sum of squares representation of the Hamiltonian

Express or approximate $v \mapsto \mathcal{H}_p(v)$ in the form

$$H(v) = \sum_{1 \leq i \leq I} \alpha_i \max\{0, \langle v, e_i \rangle\}^2 + \sum_{1 \leq j \leq J} \beta_j \langle v, f_j \rangle^2,$$

where $e_i, f_j \in \mathbb{Z}^d$, $\alpha_i, \beta_j \geq 0$. Or more generally in the form

$$H(v) = H_0(v) + \max_{1 \leq k \leq K} H_k(v).$$

where $H_0, \cdots, H_K$ are as above.

Upwind differences discretization

Approximate $H(\nabla u(p))$ by inserting

$$\max\{0, \langle \nabla u(p), e_i \rangle\} \approx h^{-1} \max\{0, U(p) - U(p - he_i)\}$$

$$|\langle \nabla u(p), e_i \rangle| \approx h^{-1} \max\{0, U(p) - U(p - he_i), U(p) - U(p + he_i)\}$$

Riemannian hamiltonians and Voronoi's reduction

- Voronoi introduced the following polytope $\mathcal{P}$ and linear program $\mathcal{L}(D)$

$$\mathcal{P} := \{M \in S_d^{++}; \forall e \in \mathbb{Z}^d, \langle e, Me\rangle \geq 1\},$$
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- Kuhn-Tucker optimality conditions: there exists $(\lambda_i, e_i) \in (\mathbb{R}_+ \times \mathbb{Z}^d)^{d'}$, where $d' = d(d+1)/2$, such that

$$D = \sum_{1 \leq i \leq d'} \lambda_i e_i \otimes e_i.$$

- Represents the Riemannian hamiltonian

$$H(v) := \langle v, Dv \rangle = \sum_{1 \leq i \leq d'} \lambda_i \langle v, e_i \rangle^2$$

Curvature penalized shortest paths

Define the cost of a unit speed curve $\gamma : [0, T] \to U$, with curvature κ, as

$$\int_0^T \mathcal{C}(\kappa(t)) \frac{\mathrm{d}t}{s(\gamma(t))}$$

We consider three curvature costs. PDE $\mathcal{H}(\nabla u) = s$, posed on the lifted domain $\Omega = U \times \mathbb{S}^1$, with points $p = (x, \theta)$.

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- Euler elastica model $\mathcal{C}(\kappa) := 1 + \kappa^2$
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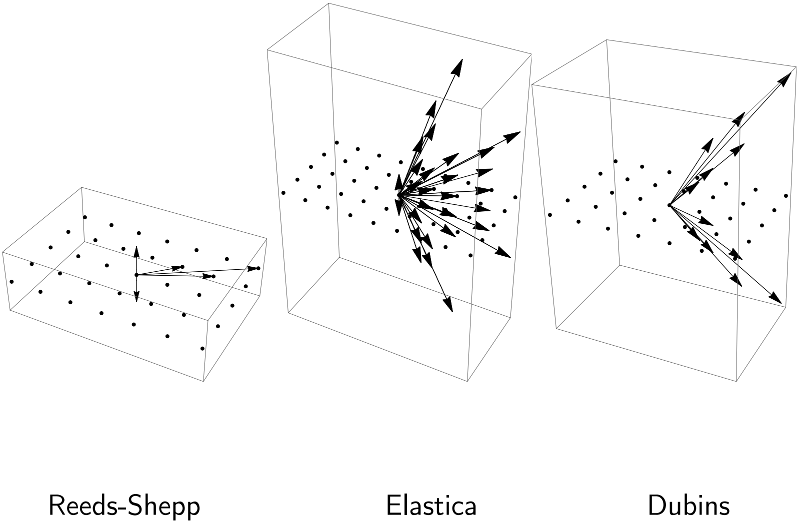
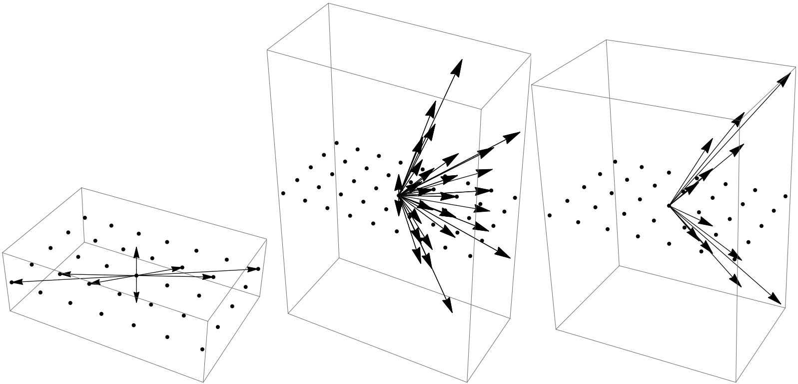
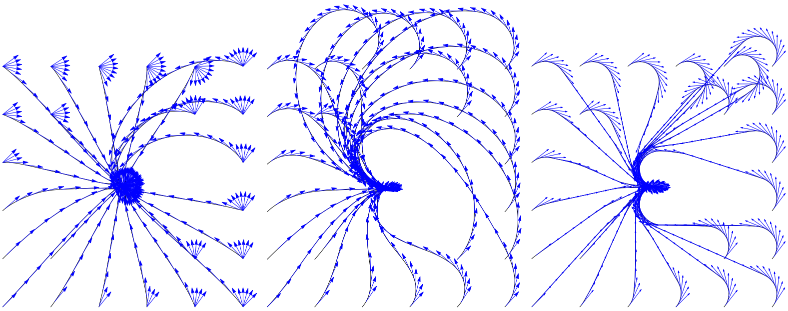

Reeds-Shepp
Elastica
Dubins



Reeds-Shepp (rev. gear) Elastica Dubins

Qualitative features of the models



Reeds-Shepp Elastica Dubins

- Reeds-Shepp's car can rotate in place (w.o. rev gear)
- Euler's car optimal paths are smooth.
- Dubin's car has a turning radius of 1.

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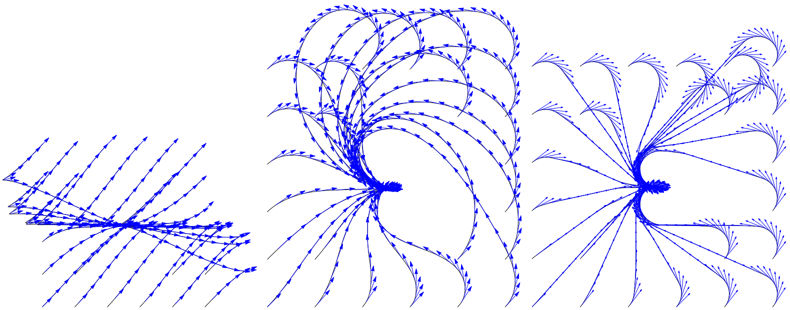

Reeds-Shepp (rev. gear) Elastica Dubins

- Reeds-Shepp's car can rotate in place (w.o. rev gear), or do cusps (with rev gear).
- Euler's car optimal paths are smooth.
- Dubin's car has a turning radius of 1.

Conclusion: Hamiltonian approach

Pros:

- Applies to a variety of metrics.
- Easy to implement.
- Cheap numerically
  (Main cost is maintaining the priority queue)

Cons:

- Hard to adapt to unstructured grids.