Geodesics and Fast Marching Methods

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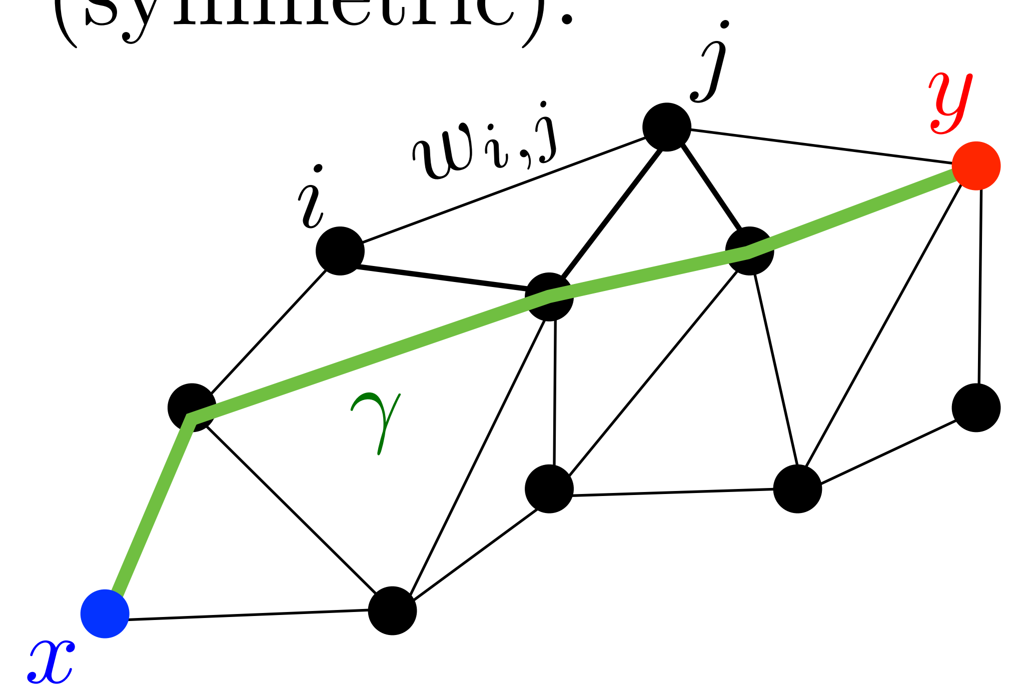
Geodesic computation on a graph

Graph: (V, E), $V = \{1, \ldots, n\}, E \subset V^2$ (symmetric).

Cost: $(w_{i,j})_{(i,j) \in E}$, $w_{i,j} > 0$.

Path: $\gamma = (\gamma_1, \ldots, \gamma_K), (\gamma_k, \gamma_{k+1}) \in E$.



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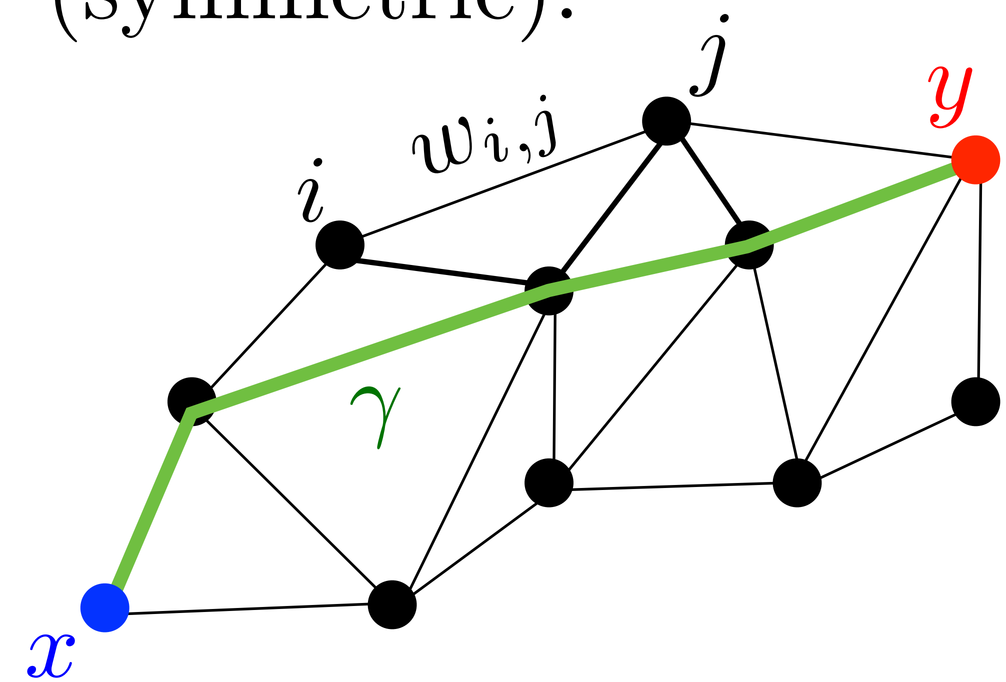

Geodesic distance:

$$d(x, y) = \min_{\gamma_1 = x, \gamma_K = y} L(\gamma).$$

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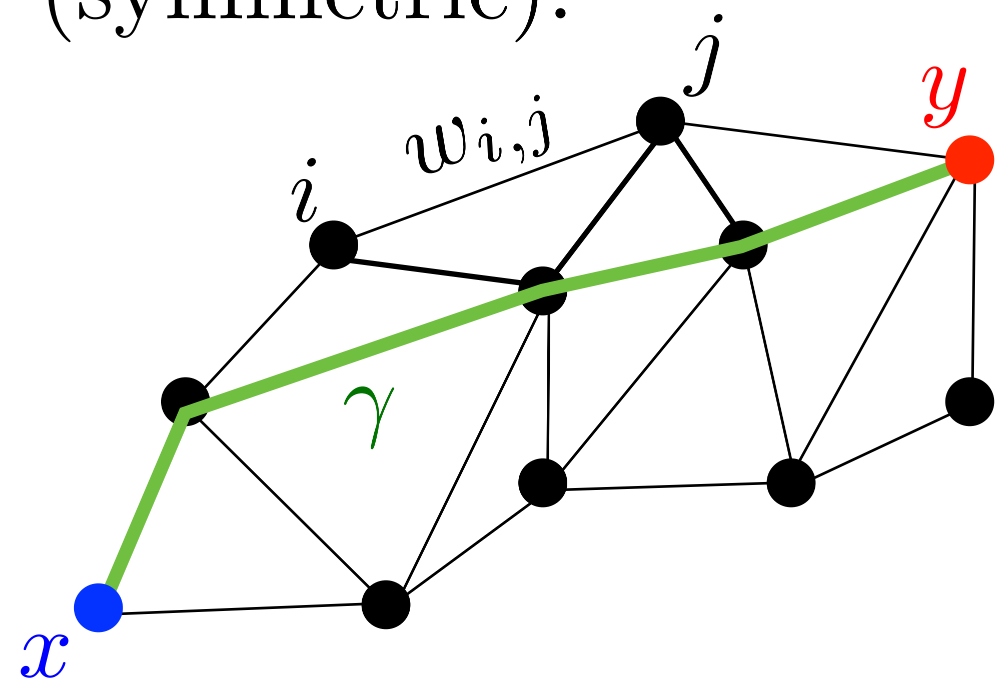

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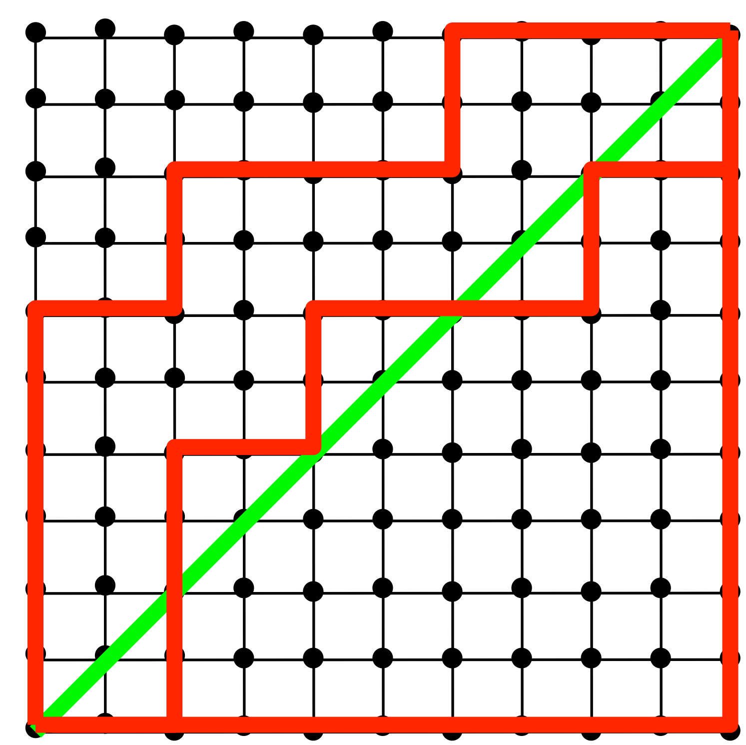
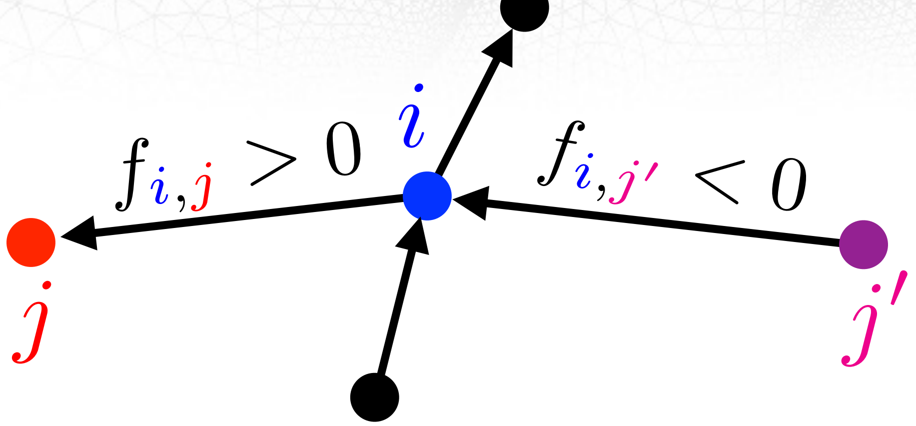
Difficulty: metrication error.



Connections with Maxflow Problems

Flow on edge: $f_{j,i} = -f_{i,j}$.

$$\text{div}(f)_i \overset{\text{def.}}{=} \sum_{j \sim i} f_{i,j}, \quad \nabla \overset{\text{def.}}{=} -\text{div}^\top$$



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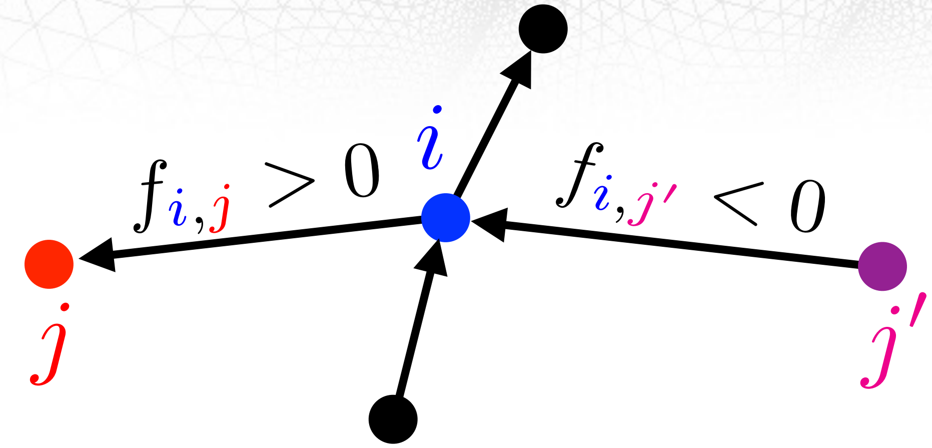
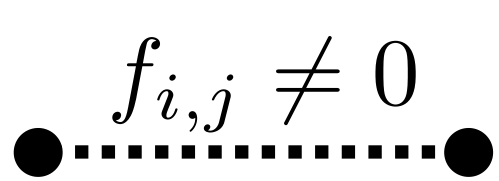
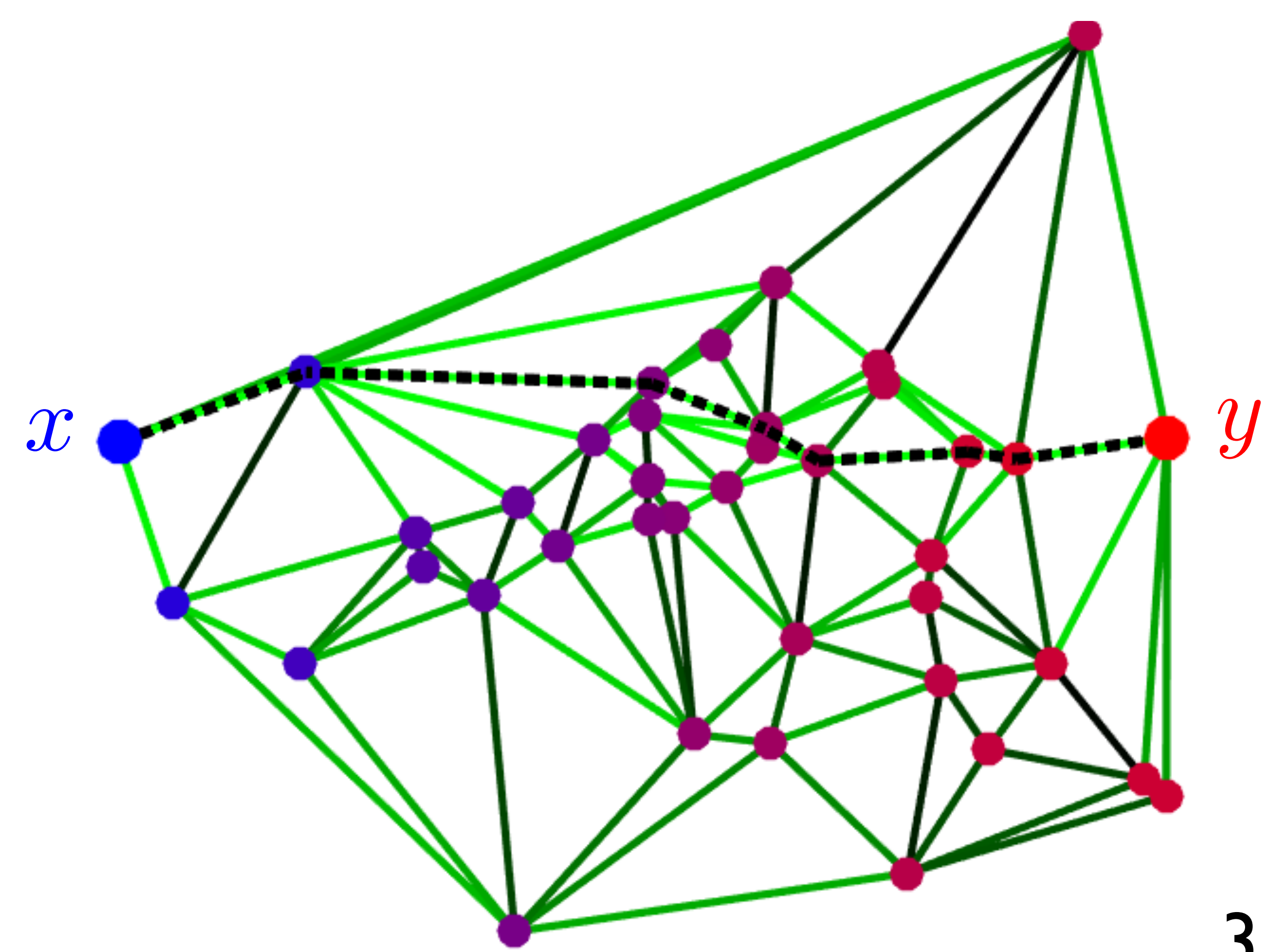


$$d(x,y) = \min_{f \in \mathbb{R}^E} \left\{ \sum_{(i,j)\in E} w_{i,j} |f_{i,j}| \; ; \; \text{div}(f) = \delta_x - \delta_y \right\}$$

$\rightarrow$ recast as max-flow.





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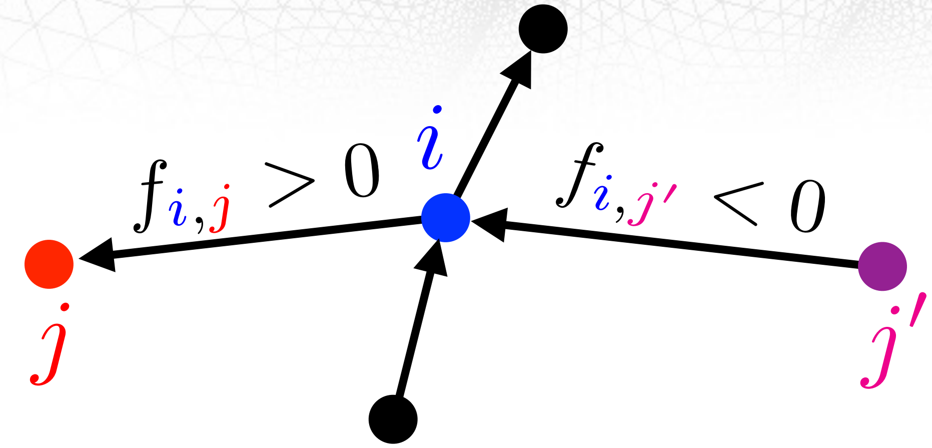

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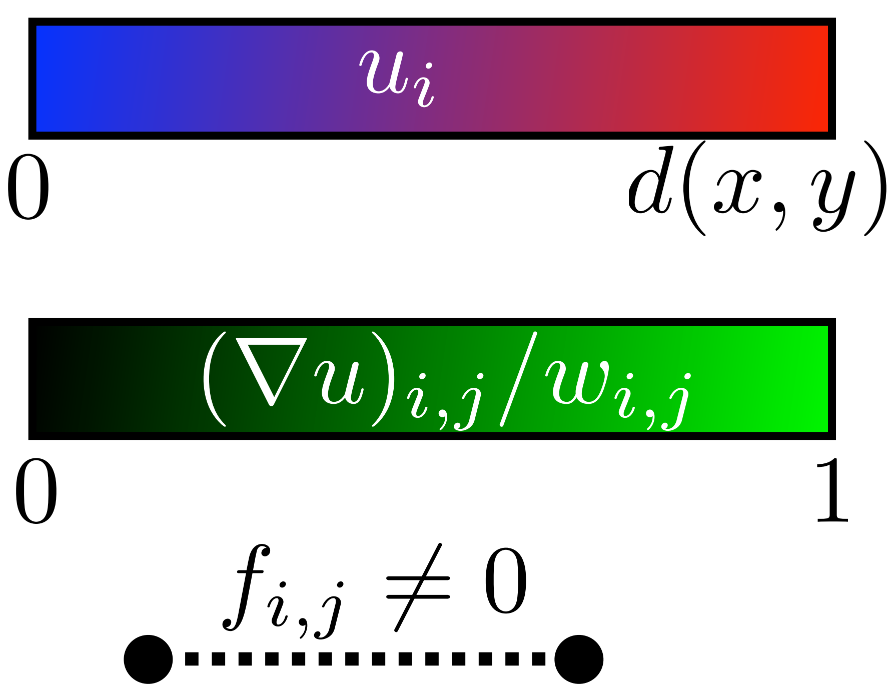
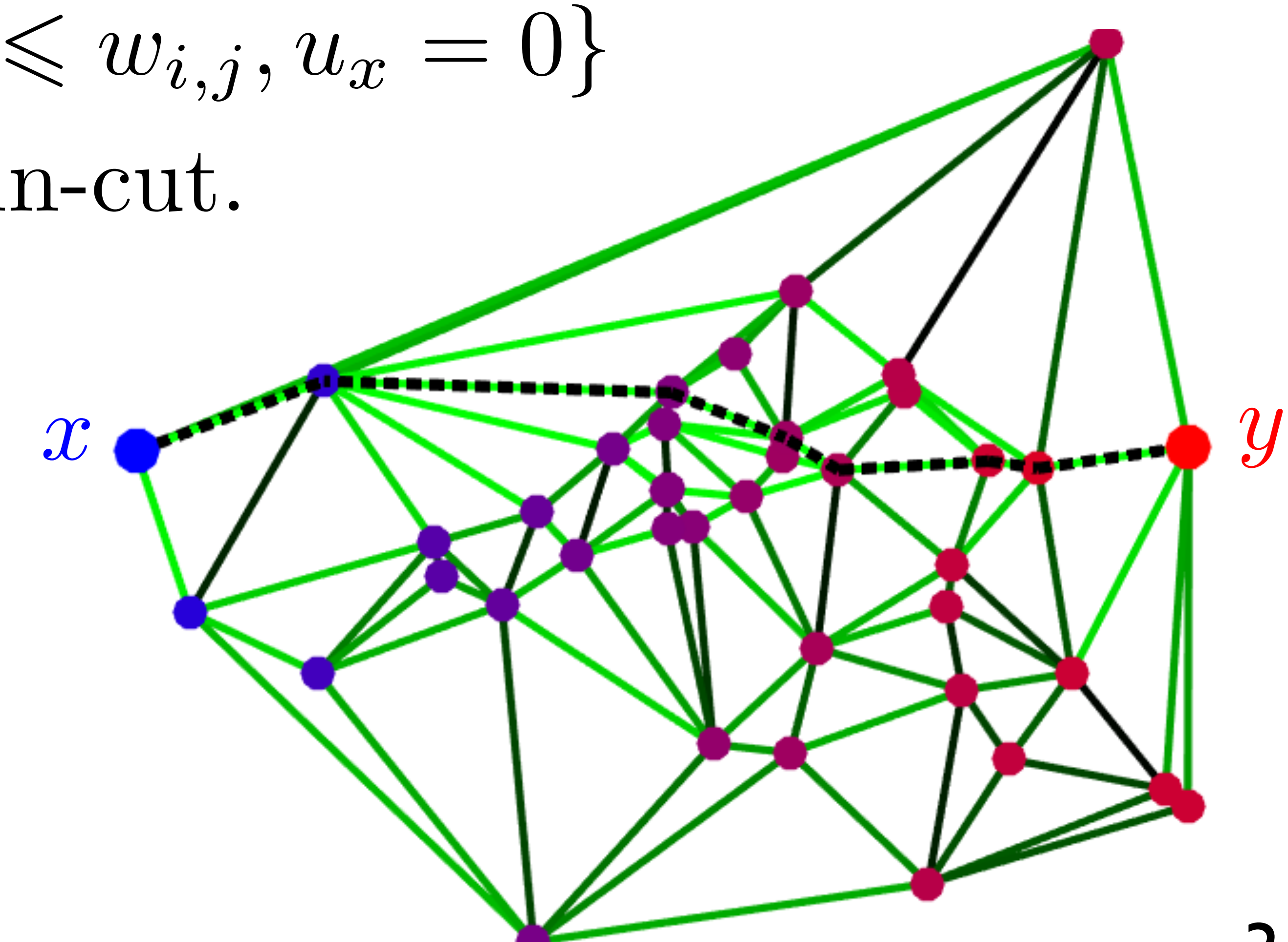
$\rightarrow$ recast as max-flow.

$$= \max_{u \in \mathbb{R}^N} \{ u_y \;;\; |(\nabla u)_{i,j}| \leqslant w_{i,j}, u_x = 0 \}$$

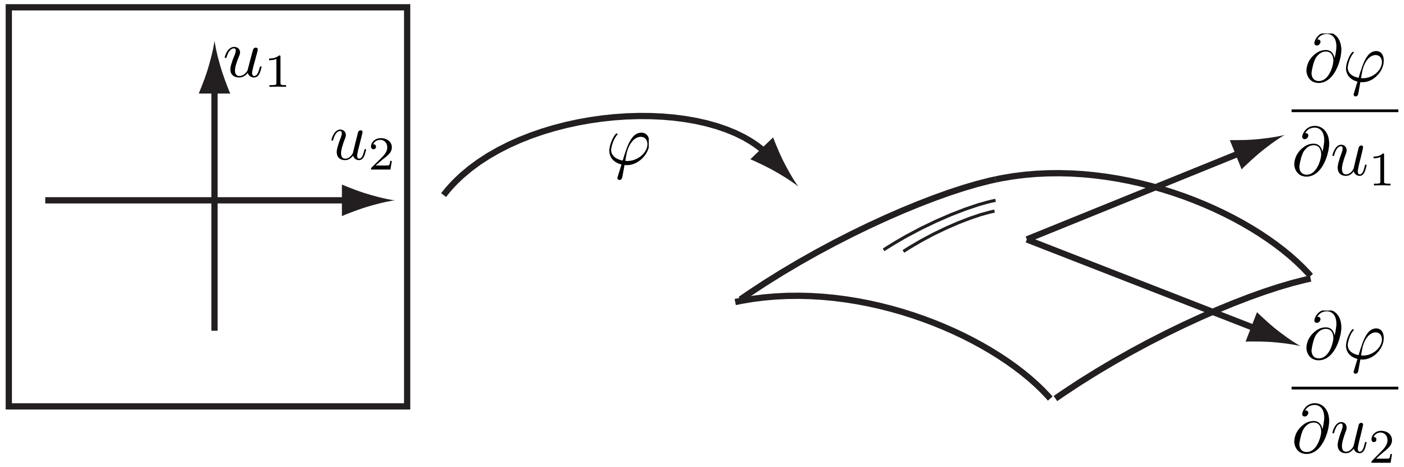
$\rightarrow$ recast as min-cut.

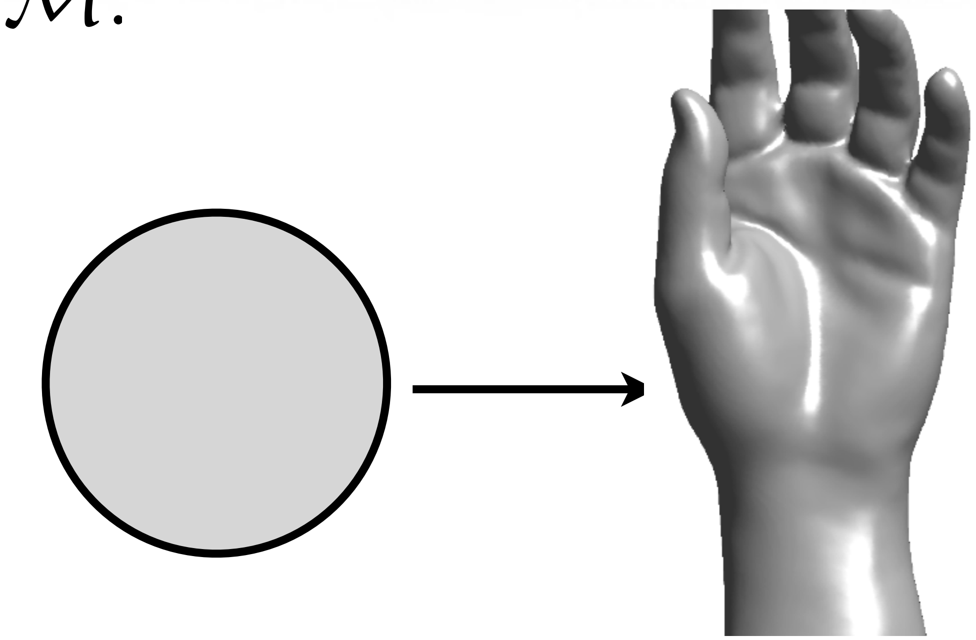




Parametric Surfaces

Parameterized surface: $u \in \mathbb{R}^2 \mapsto \varphi(u) \in \mathcal{M}$.





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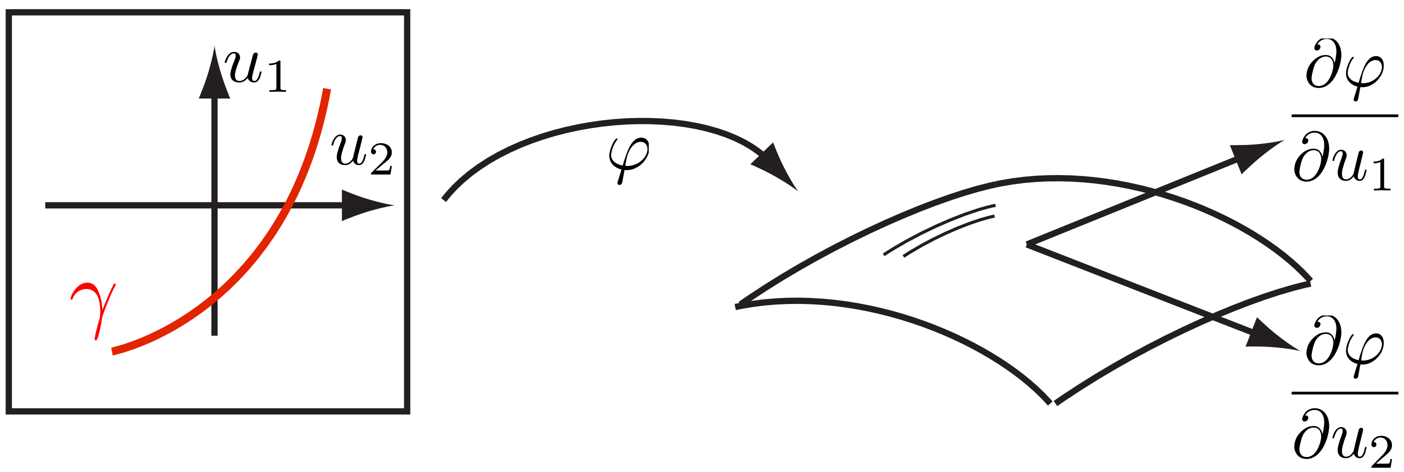

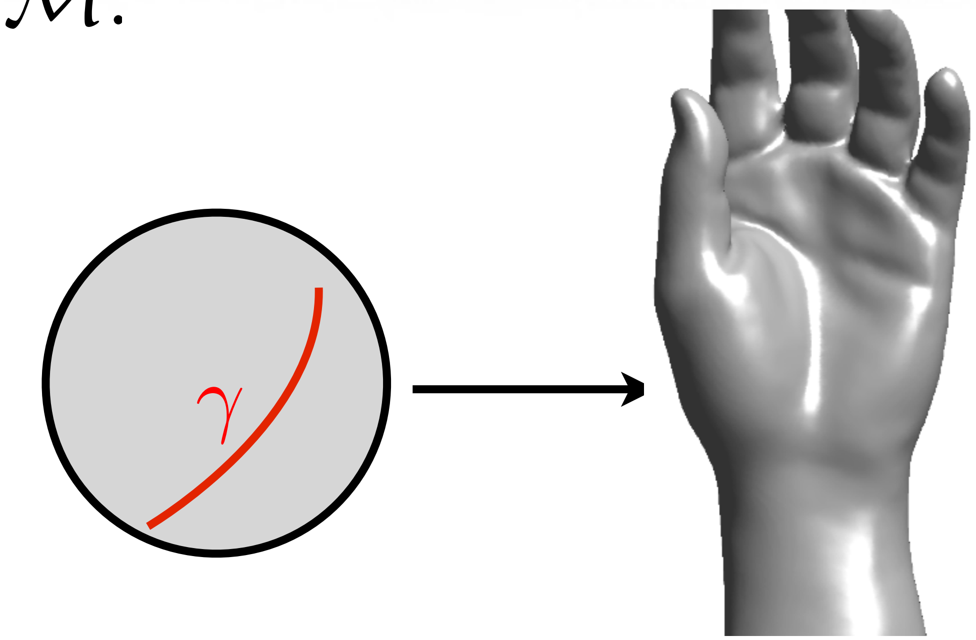

Curve in parameter domain: $t \in [0, 1] \mapsto \gamma(t) \in \mathcal{D}$.

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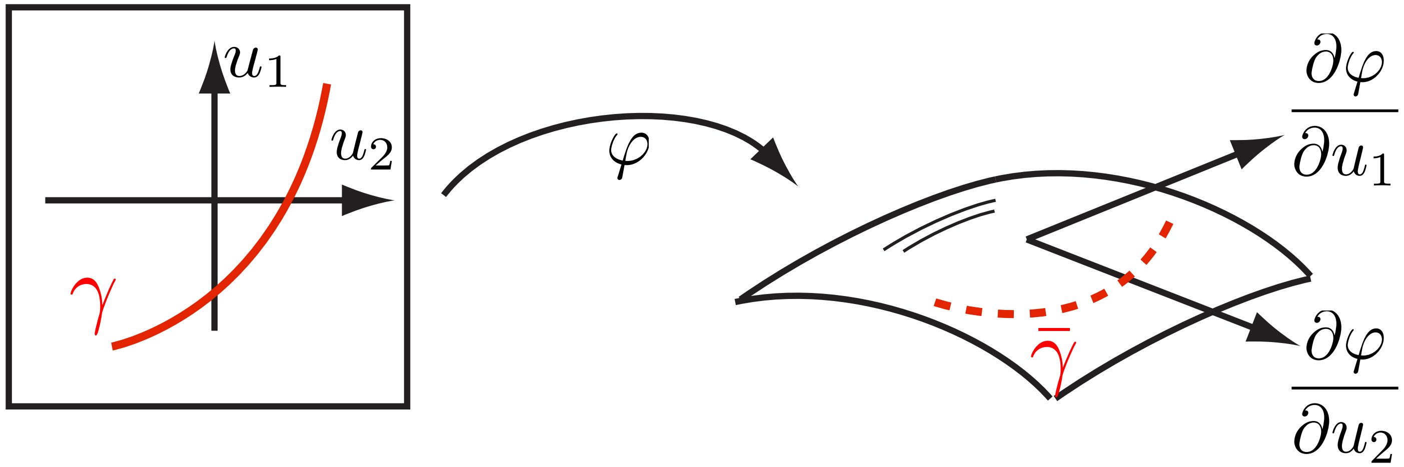

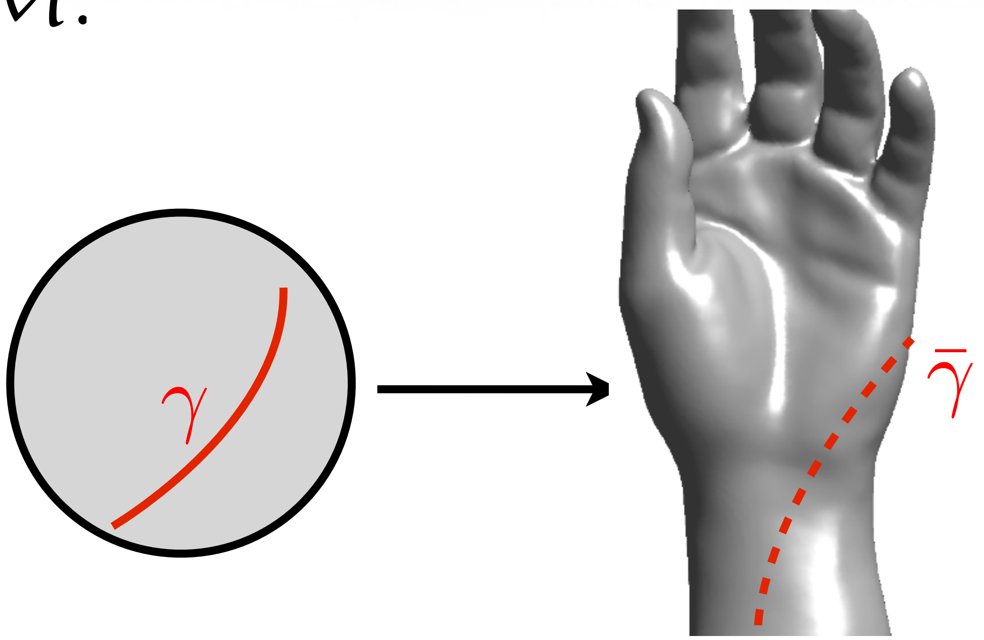

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Geometric realization: $\bar{\gamma}(t) \overset{\text{def.}}{=} \varphi(\gamma(t)) \in \mathcal{M}$.

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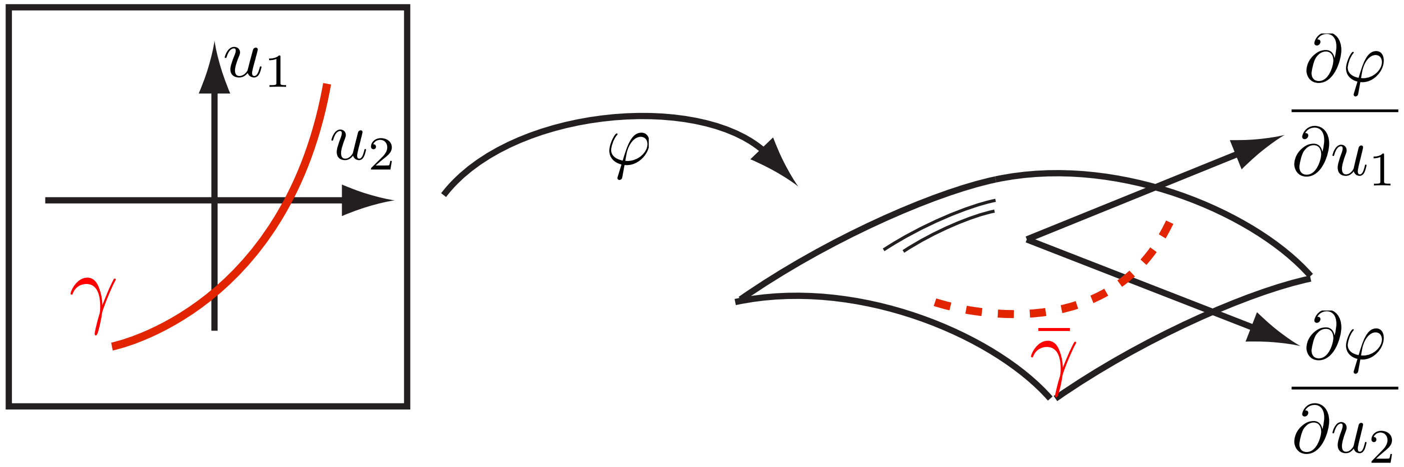

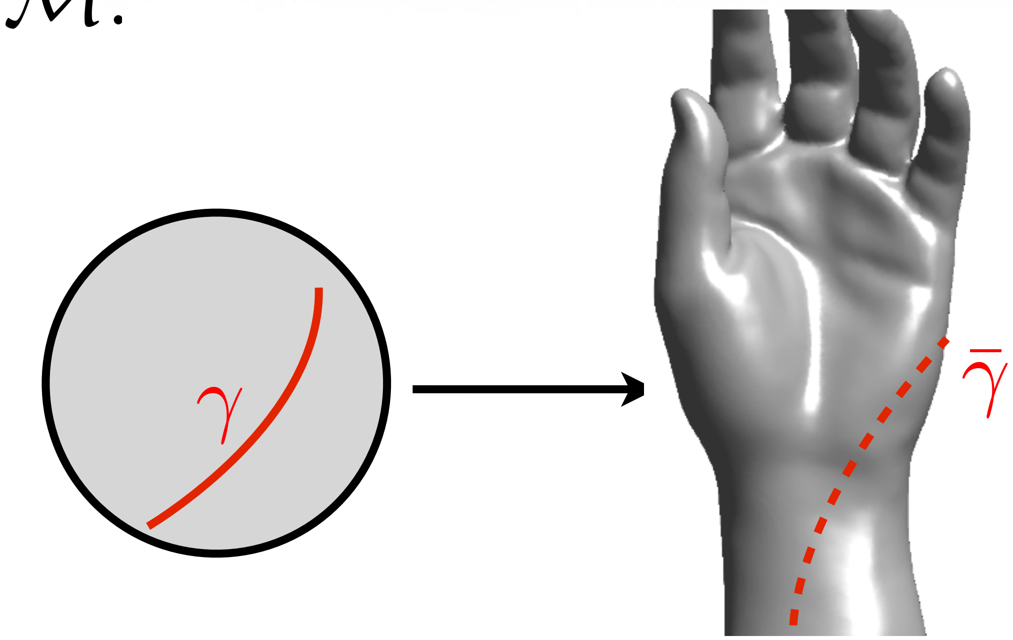

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For an embedded manifold $\mathcal{M} \subset \mathbb{R}^n$:

First fundamental form: $$I_\varphi = \left(\langle \frac{\partial \varphi}{\partial u_i}, \frac{\partial \varphi}{\partial u_j} \rangle \right)_{i,j=1,2}.$$

Length of a curve

$$L(\gamma) \overset{\text{def.}}{=} \int_0^1 \|\bar{\gamma}'(t)\| \mathrm{d}t = \int_0^1 \sqrt{\gamma'(t) I_{\gamma(t)} \gamma'(t)} \mathrm{d}t.$$

Riemannian Manifold

Riemannian manifold: $\mathcal{M} \subset \mathbb{R}^n$ (locally)

Riemannian metric: $H(x) \in \mathbb{R}^{n \times n}$, symmetric, positive definite.

Length of a curve $\gamma(t) \in \mathcal{M}$: $L(\gamma) \stackrel{\text{def.}}{=} \int_0^1 \sqrt{\gamma'(t)^{\mathrm{T}} H(\gamma(t))\gamma'(t)}\mathrm{d}t.$

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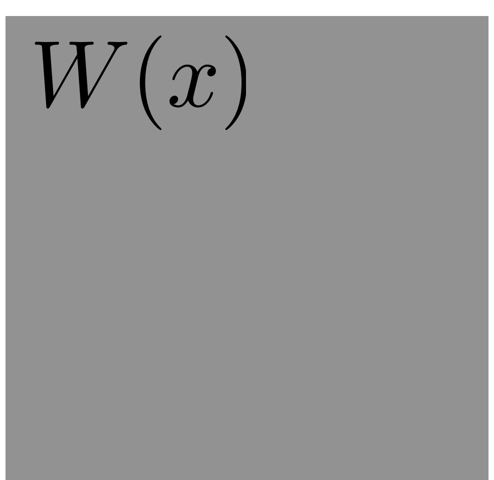

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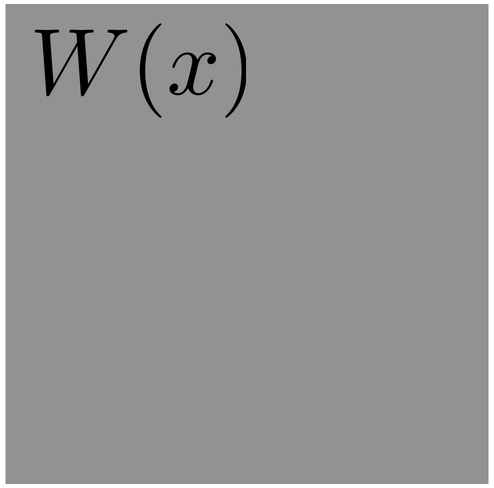

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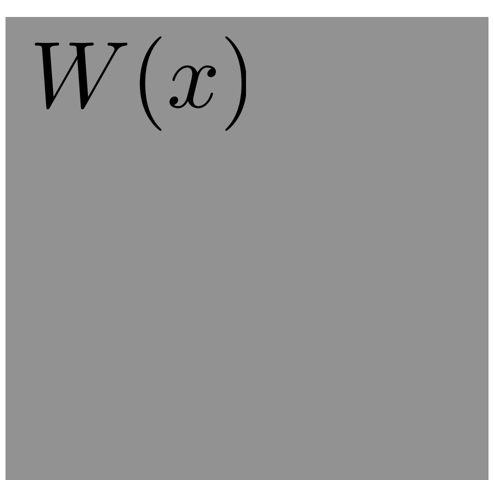

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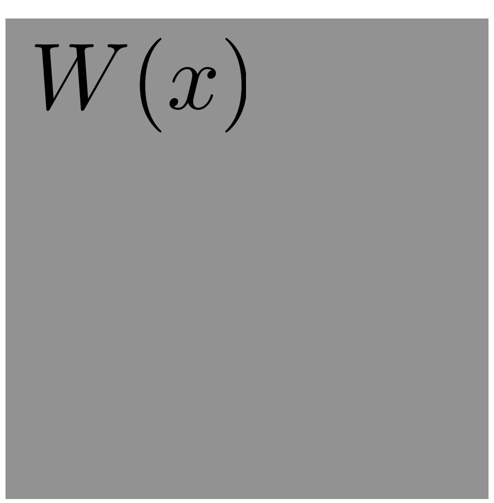
- *Image processing:* image I, $W(x)^2 = (\varepsilon + \|\nabla I(x)\|)^{-1}$.



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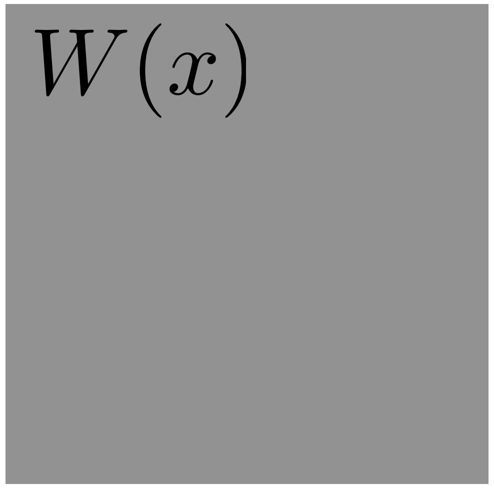

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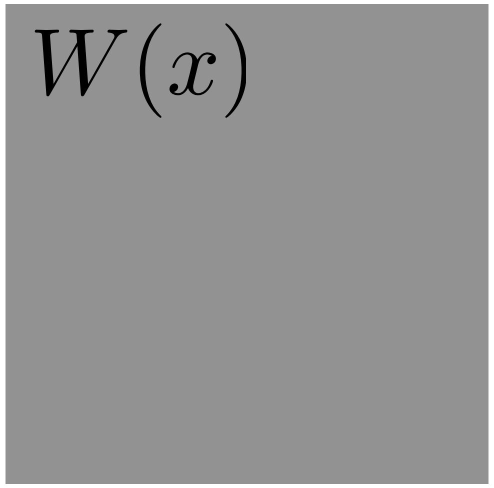
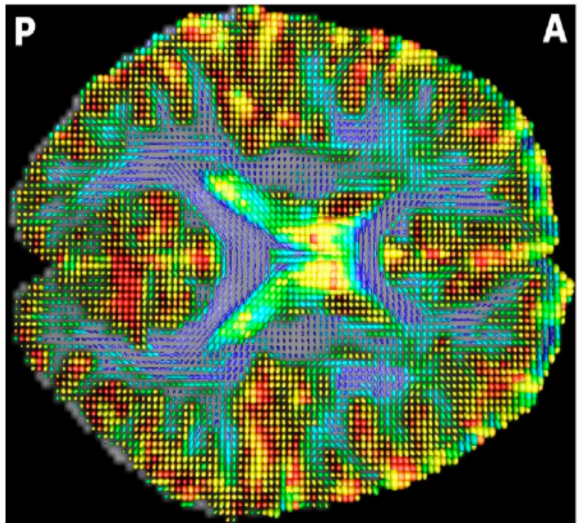
DTI imaging: $\mathcal{M} = [0,1]^3$, $H(x)$=diffusion tensor.





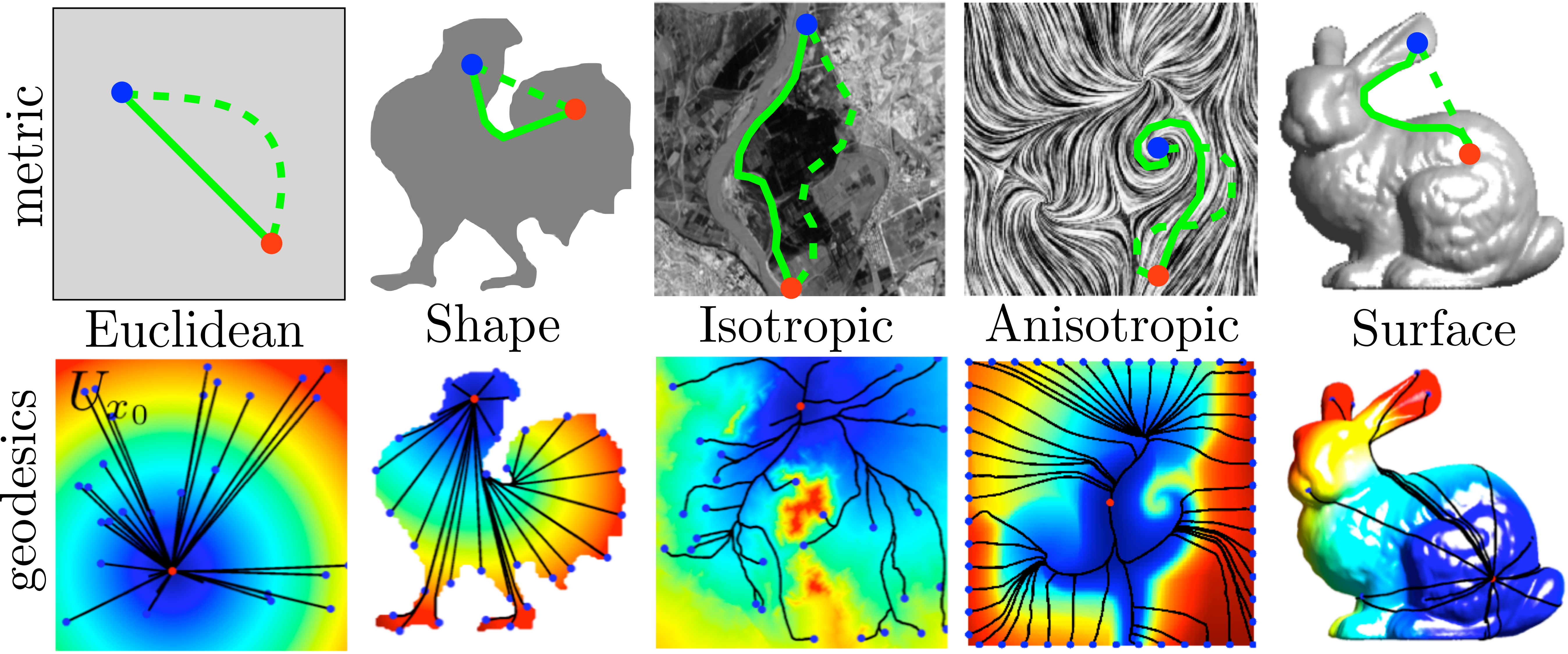
Geodesic Distances

Geodesic distance metric over $\mathcal{M} \subset \mathbb{R}^n$

$$d_{\mathcal{M}}(x, y) = \min_{\gamma(0)=x, \gamma(1)=y} L(\gamma)$$

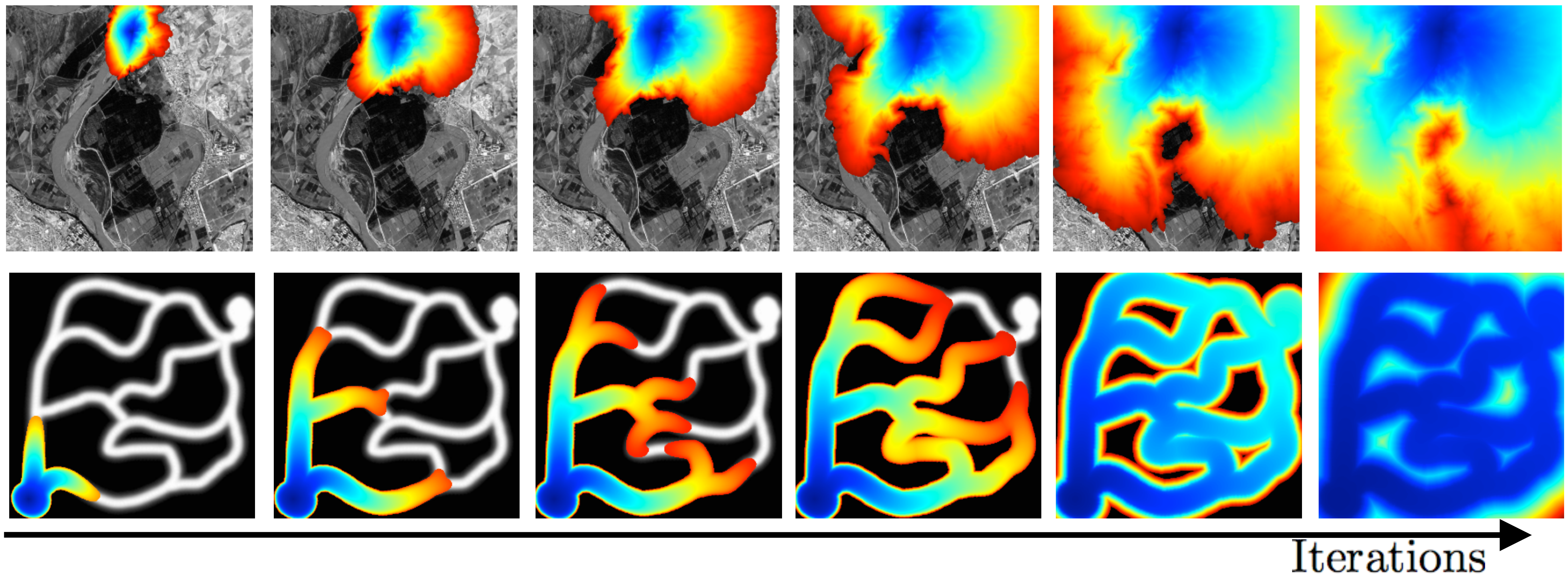
Geodesic curve: $\gamma(t)$ such that $L(\gamma) = d_{\mathcal{M}}(x, y)$.

Distance map to a starting point $x_0 \in \mathcal{M}$: $U_{x_0}(x) \stackrel{\text{def.}}{=} d_{\mathcal{M}}(x_0, x)$.



What's Next?

Laurent Cohen: Dijkstra and Fast Marching algorithms.



Jean-Marie Mirebeau: anisotropy and adaptive stencils.