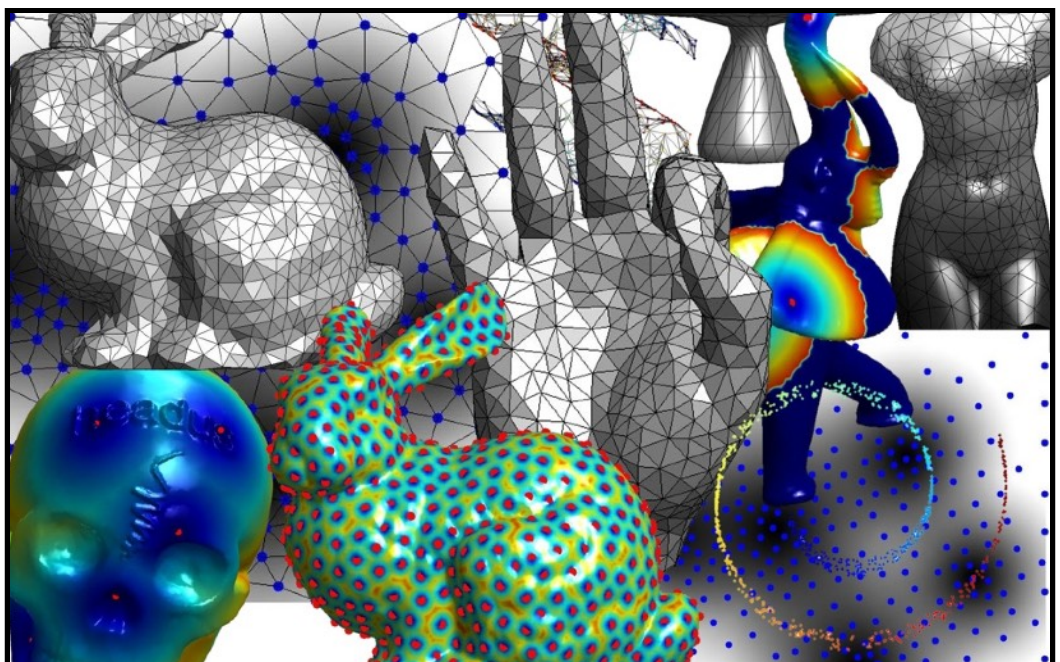
Mesh Processing Meets Graph Theory

Gabriel Peyré



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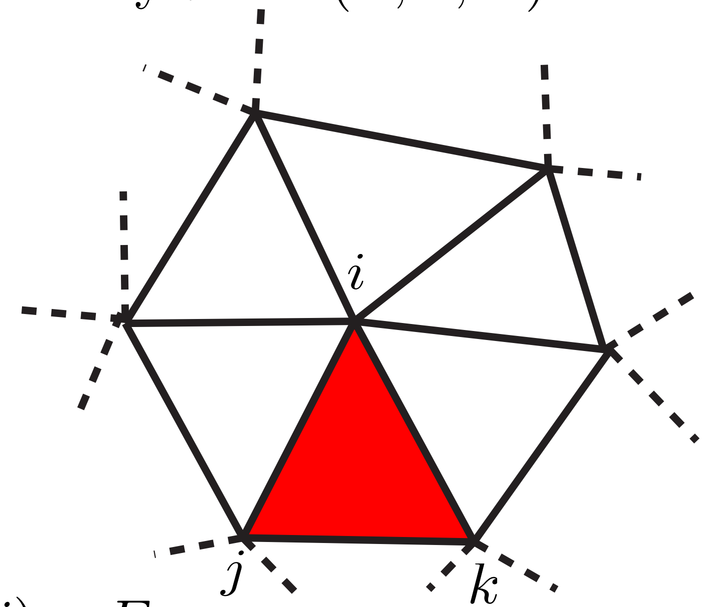
Overview

- **Triangulated Meshes**
- Operators on Meshes
- Denoising by Diffusion
- Fourier on Meshes

Triangular Meshes

Triangulated mesh: topology $M = (V, E, F)$ and geometry $\mathcal{M} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$.



Topology M**:**

- (0D) Vertices: $V \simeq \{1, \ldots, n\}$.
- (1D) Edges: $E \subset V \times V$.

  Symmetric: $(i, j) \in E \Leftrightarrow i \sim j \Leftrightarrow (j, i) \in E$.
- (2D) Faces: $F \subset V \times V \times V$.

  Compatibility: $(i, j, k) \in F \Leftarrow (i, j), (j, k), (k, i) \in E$.

$$\forall (i, j) \in E, \quad \exists (i, j, k) \in F.$$

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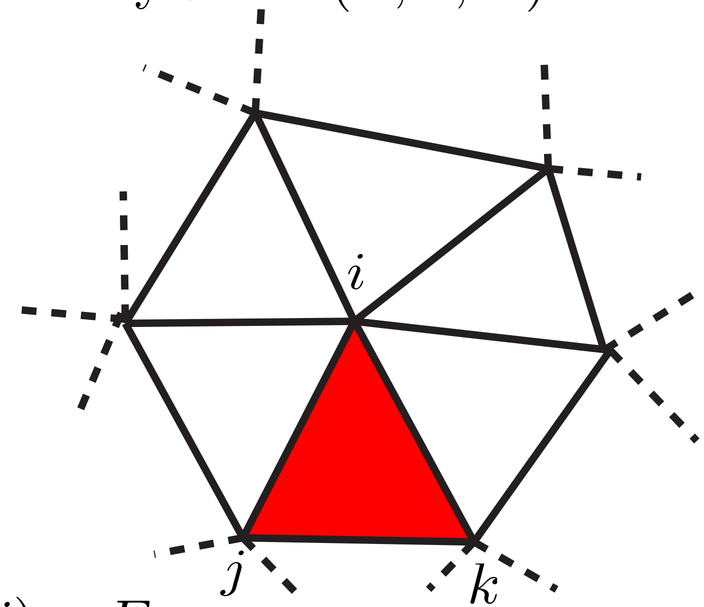

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Geometric realization $\mathcal{M}$: $\forall\, i \in V,\ x_i \in \mathbb{R}^3, \qquad \mathcal{V} \overset{\text{def.}}{=} \{x_i \setminus i \in V\}$.

Piecewise linear mesh:

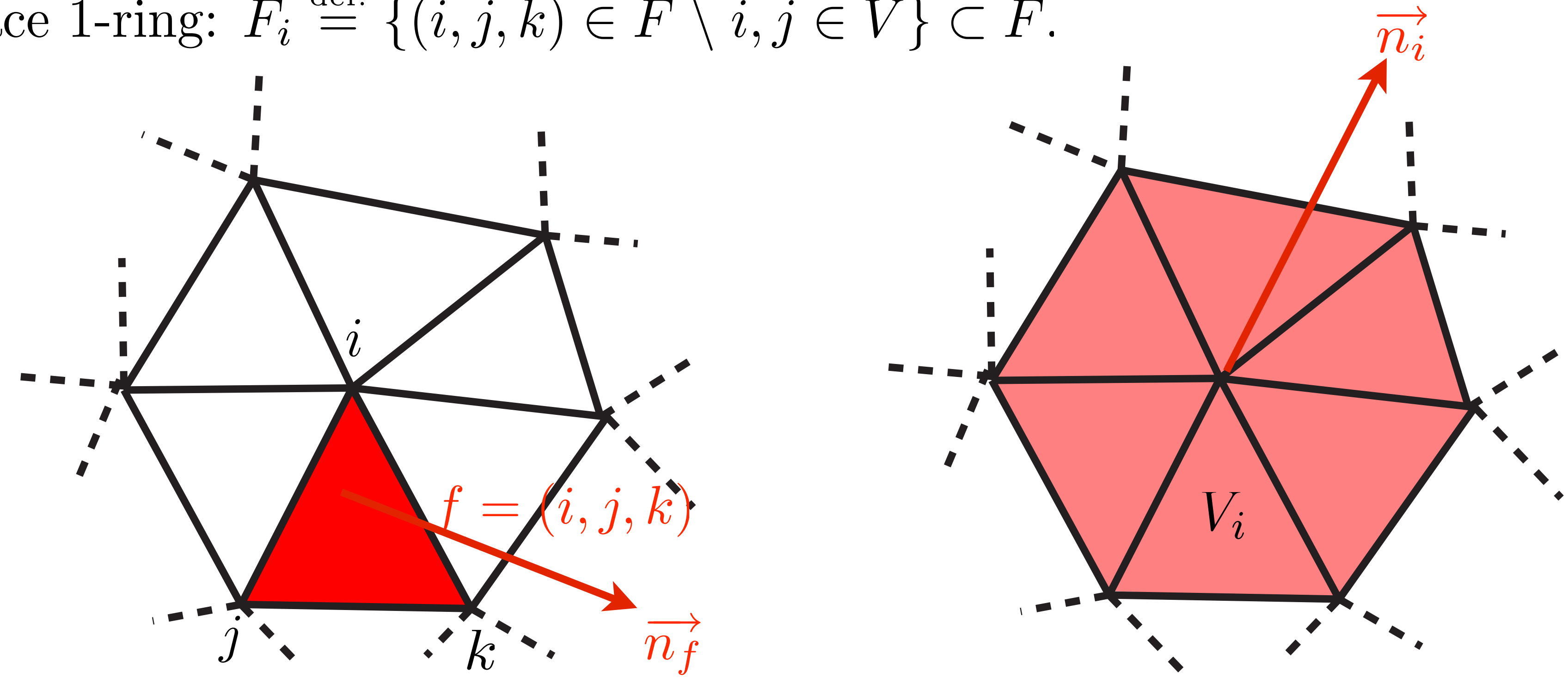
$$\mathcal{E} \overset{\text{def.}}{=} \bigcup_{(i,j) \in E} \text{Conv}(x_i, x_j) \subset \mathbb{R}^3.$$

$$\mathcal{F} \overset{\text{def.}}{=} \bigcup_{(i,j,k) \in F} \text{Conv}(x_i, x_j, x_k) \subset \mathbb{R}^3.$$

Local Connectivity

Vertex 1-ring: $V_i \overset{\text{def.}}{=} \{j \in V \setminus (i,j) \in E\} \subset V.$

Face 1-ring: $F_i \overset{\text{def.}}{=} \{(i,j,k) \in F \setminus i,j \in V\} \subset F.$



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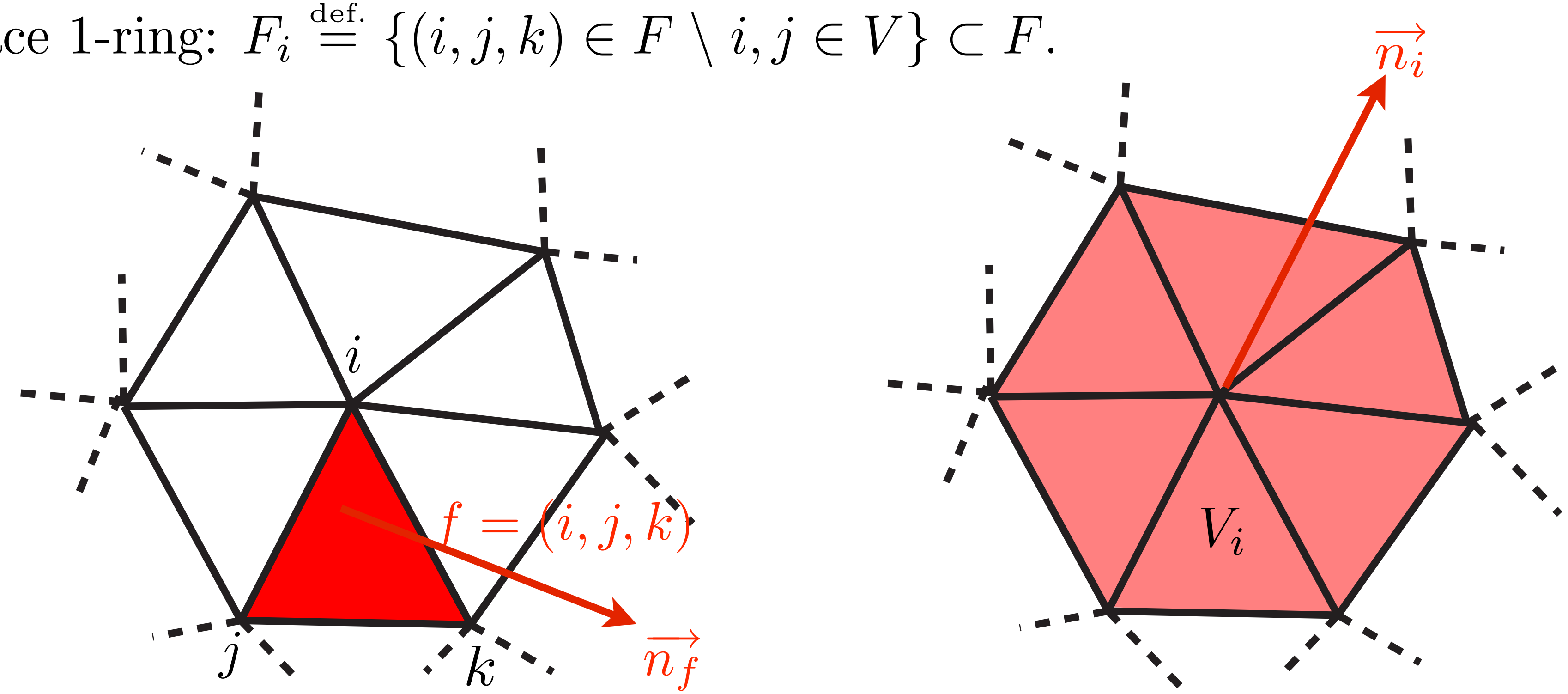
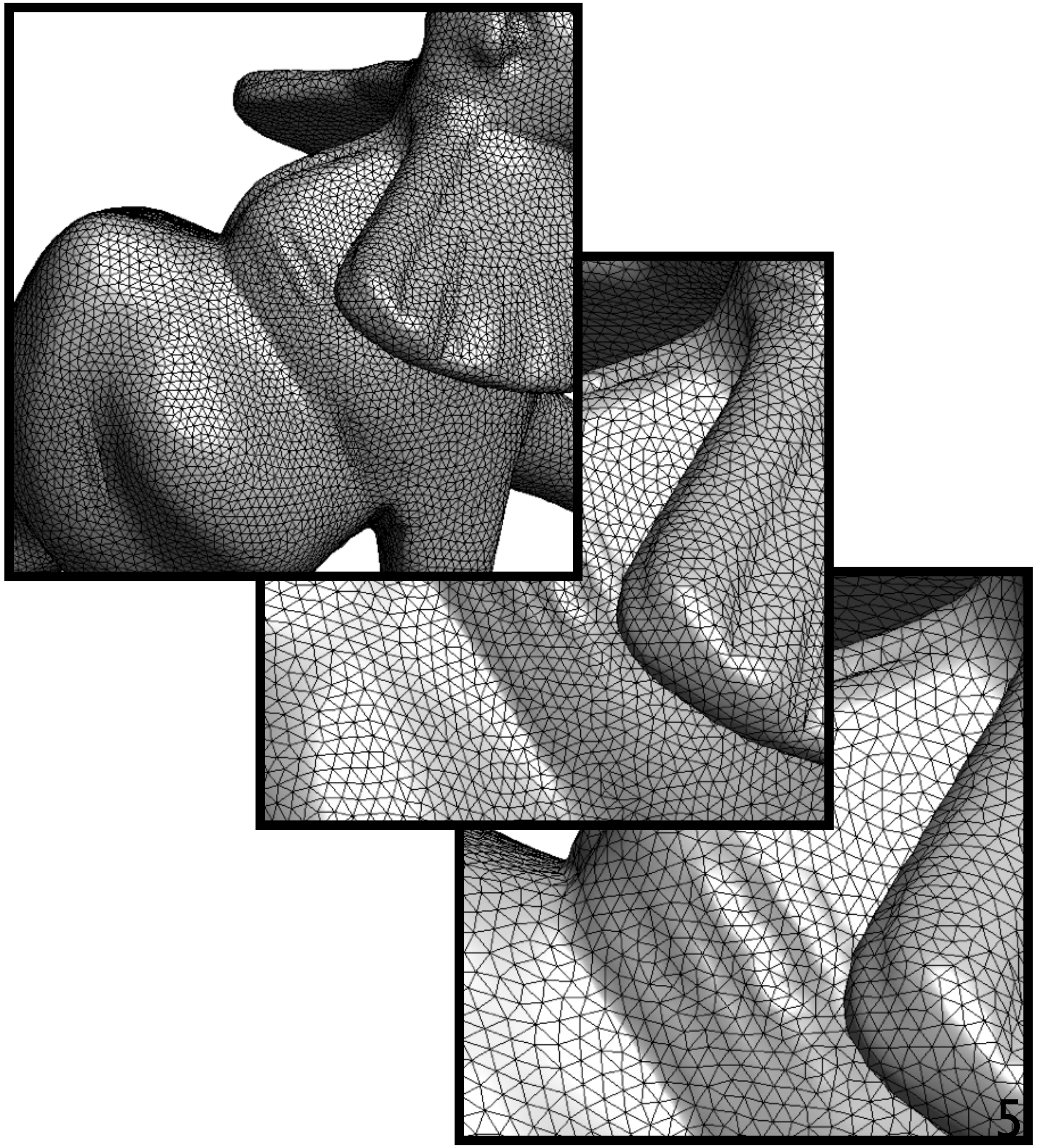


Normal Computation:

$$\forall\, f = (i,j,k) \in F, \quad \overrightarrow{n_f} \overset{\text{def.}}{=} \frac{(x_j - x_i) \wedge (x_k - x_i)}{\|(x_j - x_i) \wedge (x_k - x_i)\|}.$$

$$\forall\, i \in V, \quad \overrightarrow{n_i} \overset{\text{def.}}{=} \frac{\sum_{f \in F_i} \overrightarrow{n_f}}{\|\sum_{f \in F_i} \overrightarrow{n_f}\|}$$

Mesh Displaying



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Functions on a Mesh

Function on a mesh: $f \in \ell^2(\mathcal{V}) \simeq \ell^2(V) \simeq \mathbb{R}^n$.

$$f : \begin{cases} \mathcal{V} & \longrightarrow & \mathbb{R} \\ x_i & \longmapsto & f(x_i) \end{cases} \iff f : \begin{cases} V & \longrightarrow & \mathbb{R} \\ i & \longmapsto & f_i \end{cases} \iff f = (f_i)_{i \in V} \in \mathbb{R}^n.$$

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Inner product & norm:

$$\langle f, g \rangle \stackrel{\text{def.}}{=} \sum_{i \in V} f_i g_i \quad \text{and} \quad \|f\|^2 = \langle f, f \rangle$$

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Inner product & norm:

$$\langle f,\, g \rangle \overset{\text{def.}}{=} \sum_{i \in V} f_i g_i \quad \text{and} \quad \|f\|^2 = \langle f,\, f \rangle$$

Linear operator A:

$$A : \ell^2(V) \to \ell^2(V) \iff A = (a_{ij})_{i,j \in V} \in \mathbb{R}^{n \times n} \text{ (matrix)}.$$

$$(Af)(x_i) = \sum_{j \in V} a_{ij} f(x_j) \iff (Af)_i = \sum_{j \in V} a_{ij} f_j.$$

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Mesh processing:

– Modify functions $f \in \ell^2(V)$. $f \longrightarrow Af$

– *Example*: denoise a mesh $\mathcal{M}$ as 3 functions on M.

– *Strategy*: apply a linear operator $f \mapsto Af$.

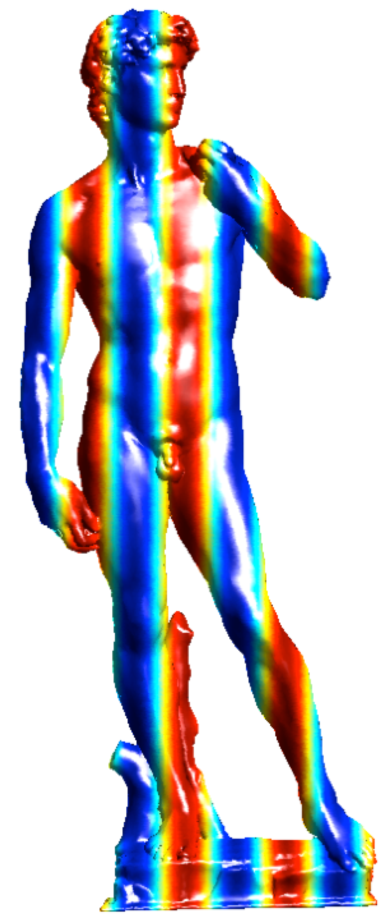
– *Remark:* A can computed from M only or from $(M, \mathcal{M})$.

Functions on Meshes

Examples:

- Coordinates: $x_i = (x_i^1, x_i^2, x_i^3) \in \mathbb{R}^3$.
- X-coordinate: $f : i \in V \mapsto x_i^1 \in \mathbb{R}$.
- Geometric mesh $\mathcal{M} \iff$ 3 functions defined on M.

$f(x_i) = x_i^1$



$f(x_i) = \cos(2\pi x_i^1)$

Local Averaging

Local operator: $W = (w_{ij})_{i,j \in V}$ where $w_{ij} = \begin{cases} > 0 & \text{if } j \in V_i, \\ 0 & \text{otherwise.} \end{cases}$

$$(Wf)_i = \sum_{(i,j) \in E} w_{ij} f_j.$$

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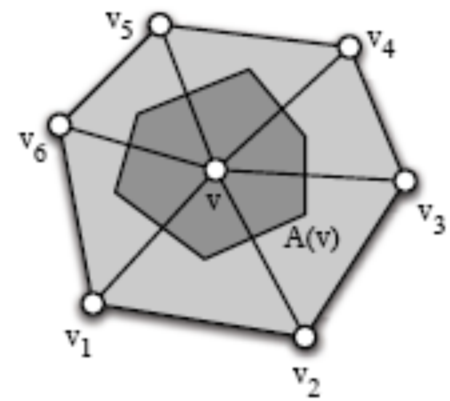

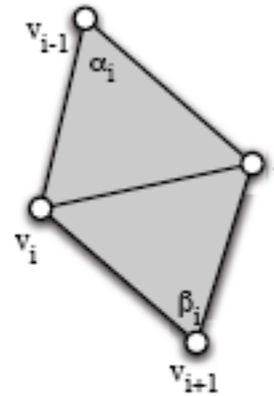

Examples: for $i \sim j$,

$w_{ij} = 1$ combinatorial

$w_{ij} = \frac{1}{\|x_j - x_i\|^2}$ distance

$w_{ij} = \cot(\alpha_{ij}) + \cot(\beta_{ij})$ conformal (explanations later)

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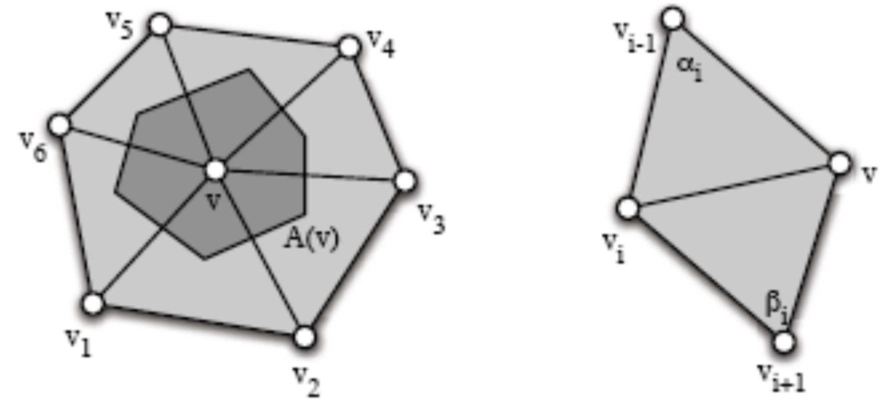

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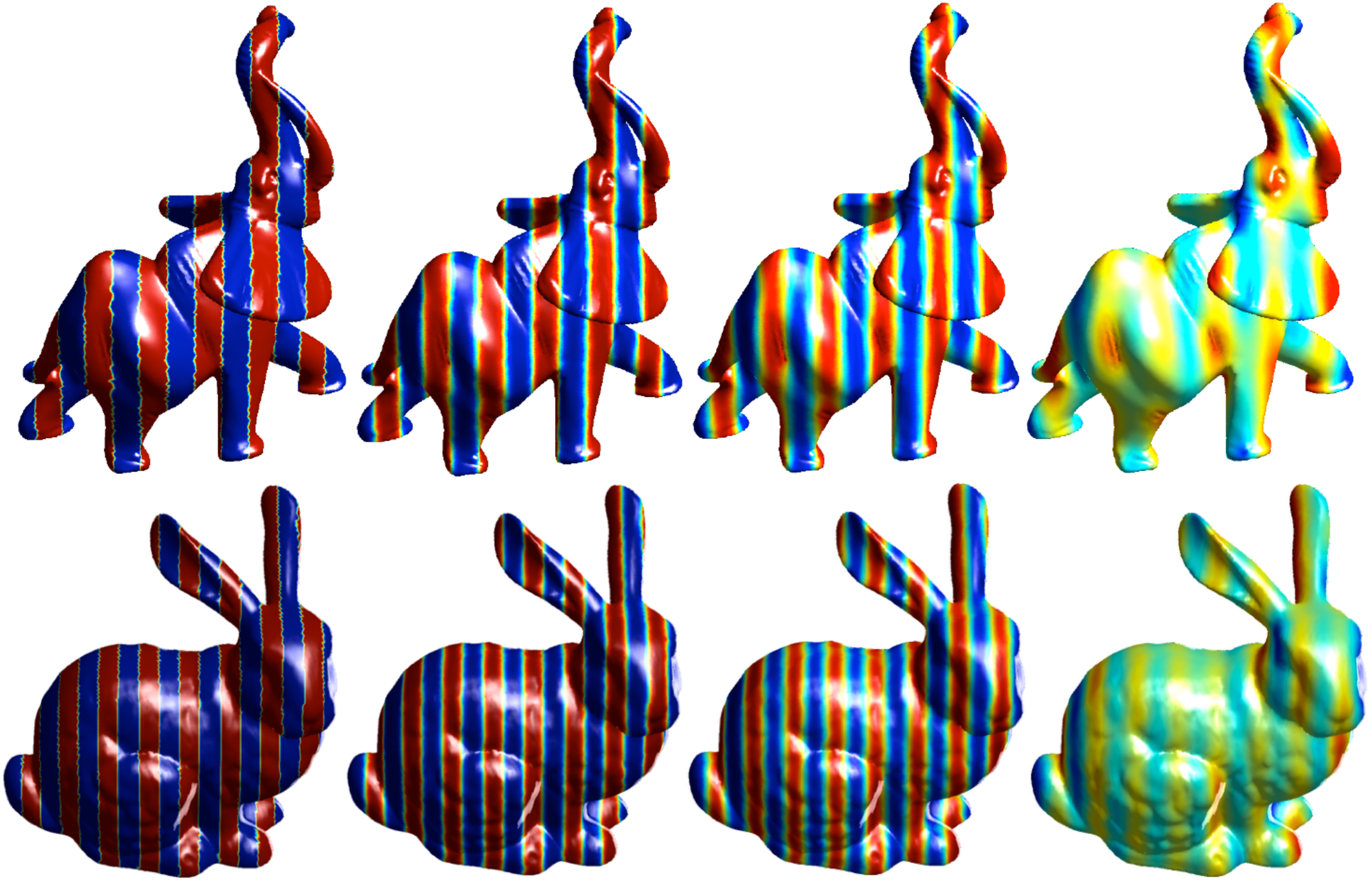
Local averaging operator $\tilde{W} = (\tilde{w}_{ij})_{i,j \in V}$: $\forall (i,j) \in E, \quad \tilde{w}_{ij} = \frac{w_{ij}}{\sum_{(i,j) \in E} w_{ij}}.$

$$\tilde{W} = D^{-1} W \quad \text{with} \quad D = \text{diag}_i(d_i) \quad \text{where} \quad d_i = \sum_{(i,j) \in E} w_{ij}.$$

Averaging: $\tilde{W} 1 = 1.$

Iterative Smoothing

Iterative smoothing: $\tilde{W}f, \tilde{W}^2, \ldots, \tilde{W}^k f$ smoothed version of f.



f $\qquad$ $\tilde{W}f$ $\qquad$ $\tilde{W}^4 f$ $\qquad$ $\tilde{W}^8 f$

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Gradient

Gradient operator: oriented edges $E_0 \overset{\text{def.}}{=} \{(i,j) \in E \setminus i < j\}$,

$$G : \ell^2(V) \longrightarrow \ell^2(E_0), \quad \Longleftrightarrow \quad G : \mathbb{R}^n \longrightarrow \mathbb{R}^p \quad \text{where} \quad p = |E_0|,$$

$$\Longleftrightarrow \quad G \in \mathbb{R}^{n \times p} \quad \text{matrix.}$$

$$\forall\, (i,j) \in E,\ i < j, \quad (Gf)_{(i,j)} \overset{\text{def.}}{=} \sqrt{w_{ij}}(f_j - f_i) \in \mathbb{R}.$$

$\rightarrow$ Derivative along direction $\overrightarrow{x_i x_j}$.

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Example: $w_{ij} = \|x_i - x_j\|^{-2}, \quad (Gf)_{(i,j)} = \frac{f(x_j)-f(x_i)}{\|x_i - x_j\|}$.

Regular grid:

– Gf discretize $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

– $G^{\mathrm{T}} v$ discretize $\operatorname{div}(v) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$.

Laplacian

$$L \overset{\text{def.}}{=} D - W, \quad \text{where} \quad D = \text{diag}_i(d_i), \quad \text{with} \quad d_i = \sum_j w_{ij}.$$

Normalized Laplacian:

$$\tilde{L} \overset{\text{def.}}{=} D^{-1/2} L D^{-1/2} = \text{Id}_n - D^{-1/2} W D^{1/2} = \text{Id}_n - D^{1/2} \tilde{W} D^{-1/2}.$$

Remarks:

- symmetric operators $L, \tilde{L} \in \mathbb{R}^{n \times n}$.
- $L1 = 0$: acts like a (second order) derivative.
- $\tilde{L}1 \neq 0$.

Theorem: $L = G^{\mathrm{T}} G \quad \text{and} \quad \tilde{L} = (G D^{-1/2})^{\mathrm{T}} (G D^{-1/2})$.

$\Longrightarrow$ L and $\tilde{L}$ are symmetric positive definite.

$$\langle Lf, f \rangle = \|Gf\|^2 = \sum_{(i,j) \in E_0} w_{ij} \|f_i - f_j\|^2$$

$$\langle \tilde{L}f, f \rangle = \|G D^{-1/2} f\|^2 = \sum_{(i,j) \in E_0} w_{ij} \left\| \frac{f_i}{\sqrt{d_i}} - \frac{f_j}{\sqrt{d_j}} \right\|^2$$

Examples of Laplacians

Example in 1D: $$(Lf)_i = \frac{1}{h^2}(2f_i - f_{i+1} - f_{i-1}) = \frac{1}{h^2} f * (-1, 2, -1)$$

$$L \xrightarrow{h \to 0} -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_i)$$

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Example in 2D:

$$(Lf)_i = \frac{1}{h^2}(4f_i - f_{j_1} - f_{j_2} - f_{j_3} - f_{j_4}) = \frac{1}{h^2} f * \begin{pmatrix} 0 & \text{-1} & 0 \\ \text{-1} & 4 & \text{-1} \\ 0 & \text{-1} & 0 \end{pmatrix}$$

$$L \xrightarrow{h \to 0} -\frac{\partial^2 f}{\partial x^2}(x_i) - \frac{\partial^2 f}{\partial y^2}(x_i) = \Delta f(x_i).$$

$L = G^{\mathrm{T}} G f$ discretize $\Delta f = \mathrm{div}(\nabla f)$.

Iterative Smoothing

Initialization: $k = 1, 2, 3,\ f_k^{(0)} = f_k.$

Iteration: $k = 1, 2, 3,\ f_k^{(s+1)} = \tilde{W} f_k^{(s)}, \quad f_k^{(s+1)}(i) = \frac{1}{|V_i|} \sum_{(j,i) \in V_i} f_k^{(s)}(i)$

Denoised: choose s, and $\tilde{x}_i = (f_1^{(s)}, f_2^{(s)}, f_3^{(s)})$

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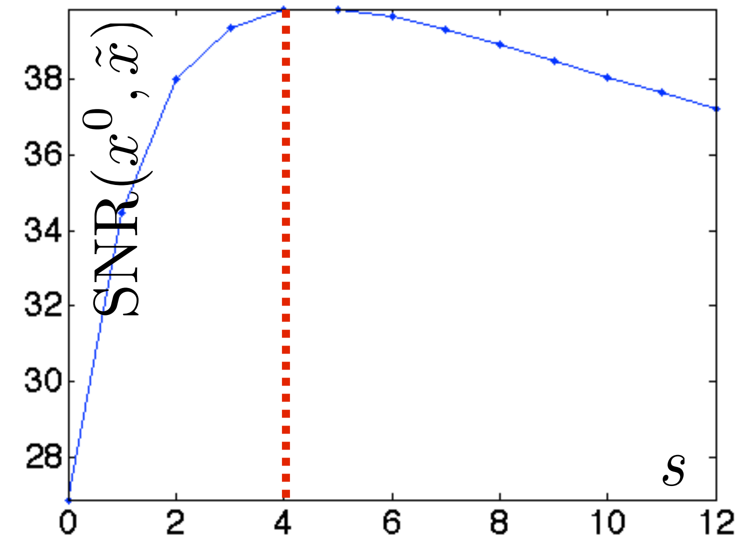

Problem: optimal choice of s

Oracle: $\max_s \mathrm{SNR}(x^0, x^{(s)}).$

Heat Diffusion

Heat diffusion: $\forall\, t > 0,\ F^{(t)} : V \to \mathbb{R}$ solving

$$\frac{\partial F^{(t)}}{\partial t} = -D^{-1} L F^{(t)} = (\mathrm{Id}_n - \tilde{W}) F^{(t)} \quad \text{and} \quad \forall\, i \in V,\ F^{(0)}(i) = f(i)$$

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Discretization: time step δ, #iterations $K \overset{\text{def.}}{=} t/\delta$.

$$\frac{1}{\delta}\left(f^{(s+1)} - f^{(s)}\right) = -D^{-1} L f^{(s)} \implies f^{(s+1)} = f^{(s)} - \delta D^{-1} L f^{(s)} = (1-\delta) f^{(s)} + \delta \tilde{W} f^{(s)}.$$

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Theorem: stable and convergent scheme if $\delta < 1$ (CFL condition)

$$f^{(t/\delta)} \quad \overset{\delta \to 0}{\longrightarrow} \quad F^{(t)}$$

$\longrightarrow$ see later for a proof.

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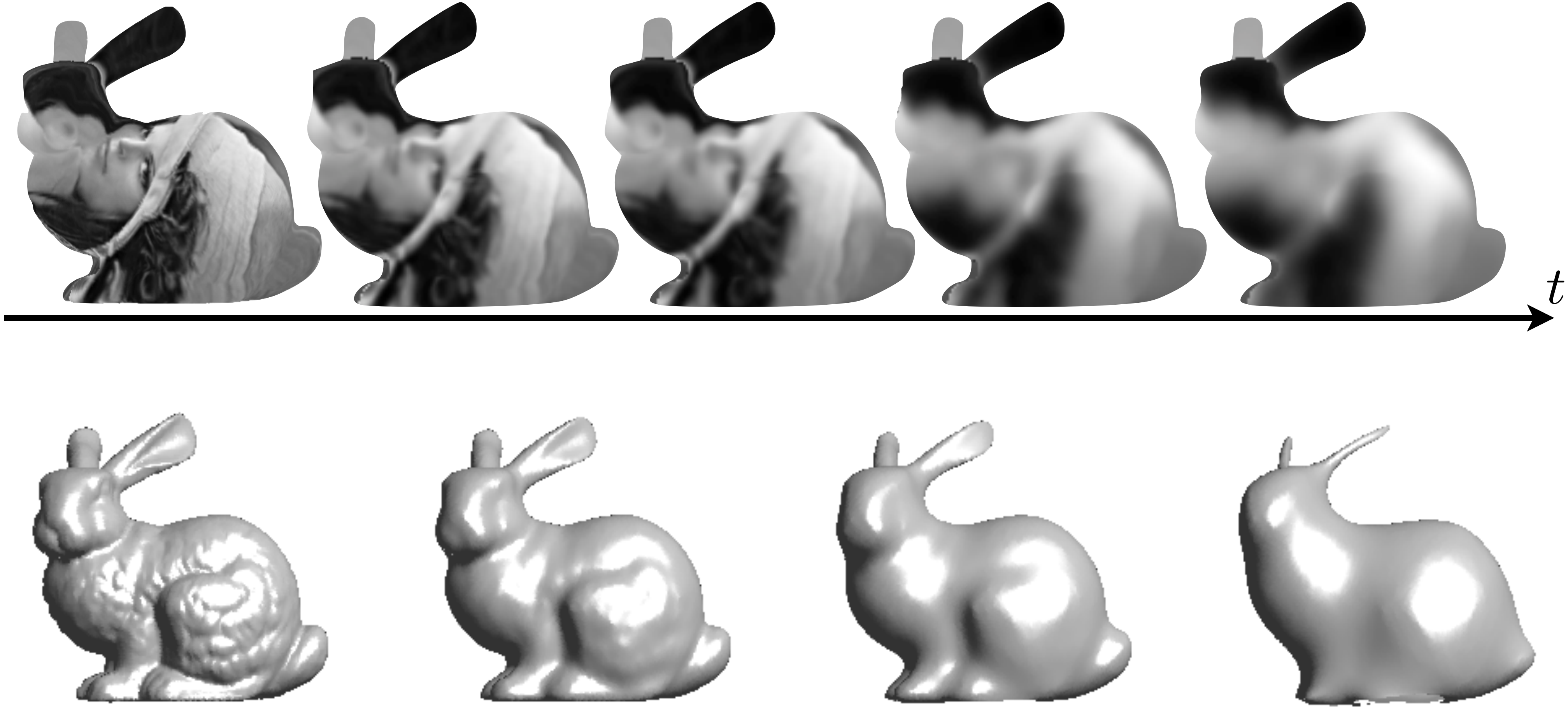
Remark: if $\delta = 1$, $f^{(s)} = \tilde{W}^k f$.

$\longrightarrow$ still stable in most cases (see later).

PDEs on Meshes

Heat diffusion: $$\frac{\partial f}{\partial t} = \Delta f \quad \text{and} \quad f(x,0) = f_0(x)$$



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Diffusion of X/Y/Z coordinates:

PDEs on Meshes

Heat diffusion: $\frac{\partial f}{\partial t} = \Delta f$ and $f(x,0) = f_0(x)$

Diffusion of X/Y/Z coordinates:

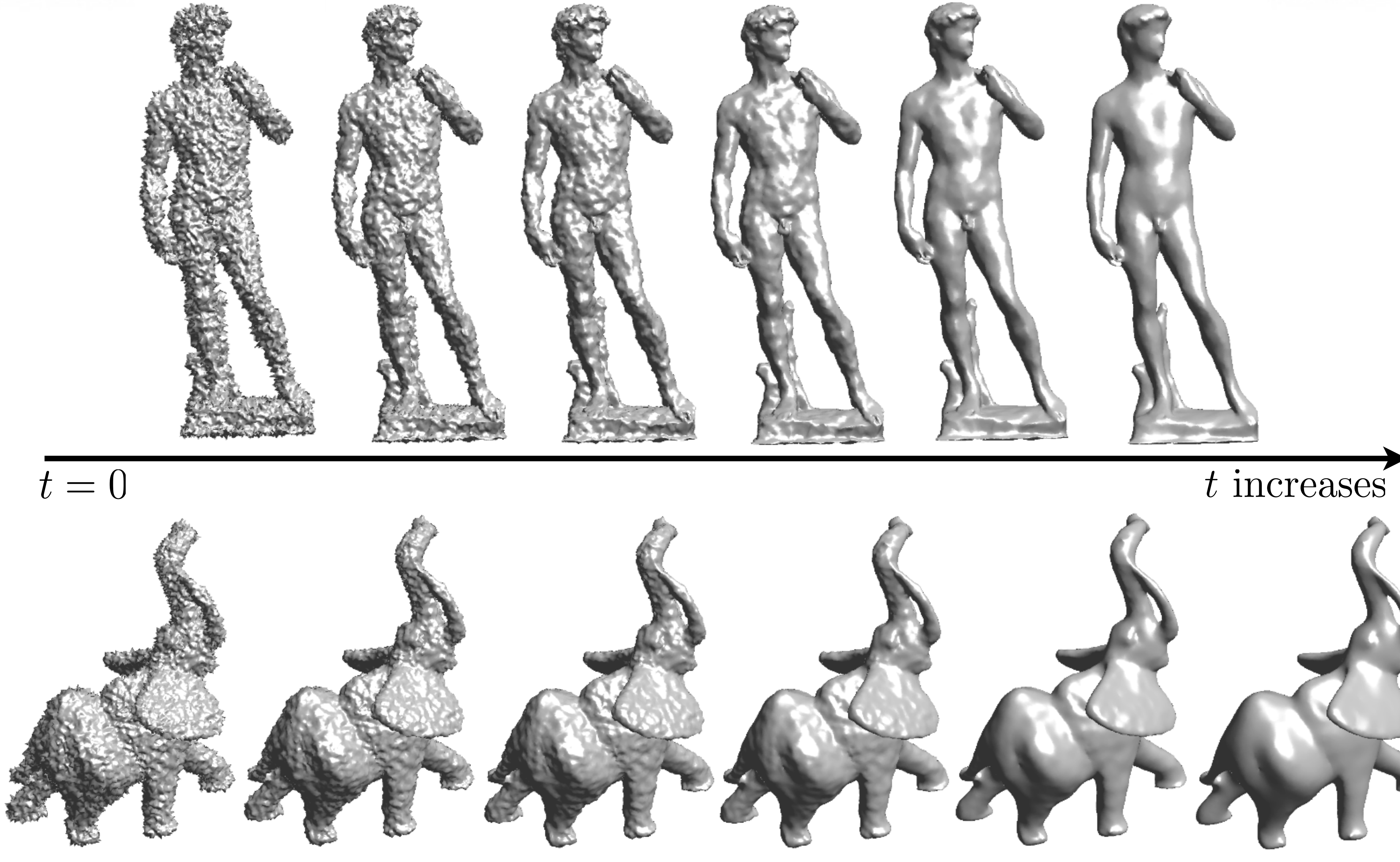
Initialization:

```
% Laplacian matrix
L=D-W;
% initialization
f1 = f;
```

Explicit Euler:

```
for i=1:3
   f1 = f1 + tau*L*f1;
end
```

Mesh Denoising with Heat Diffusion



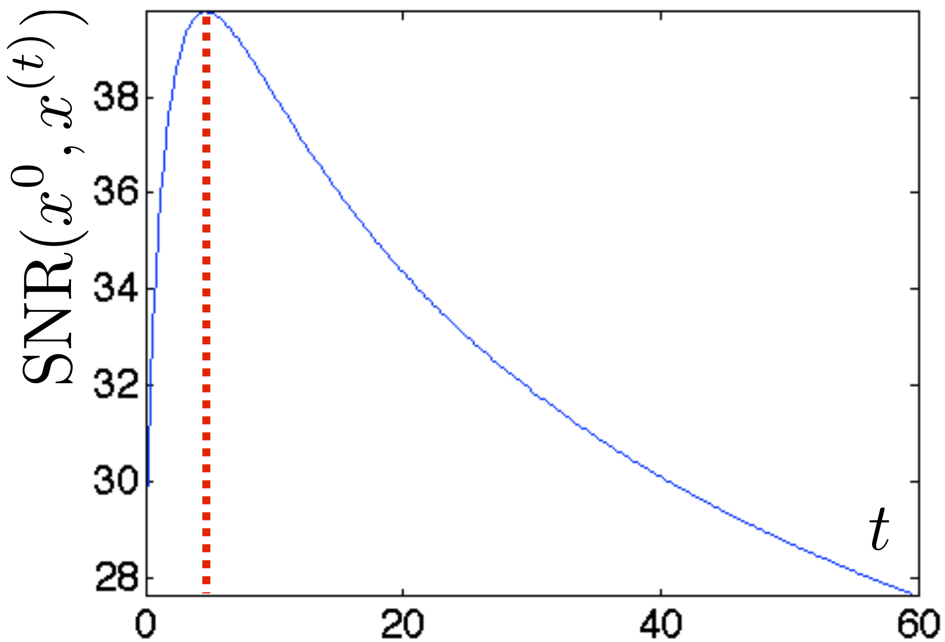
Optimal Stopping Time

Mesh: 3 functions X/Y/Z: $x_i = (f_1(i), f_2(i), f_3(i)) \in \mathbb{R}^3$.

Denoised: $x^{(t)} = (f_1^{(s)}, f_2^{(s)}, f_3^{(s)})$ for $t = s\delta$.

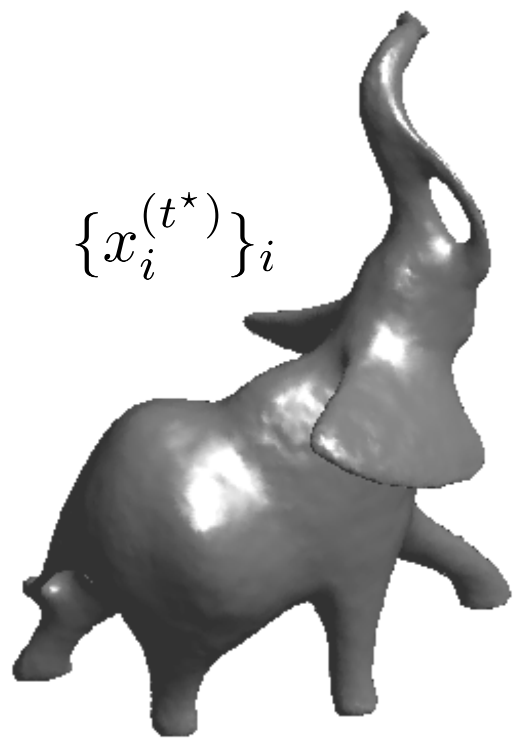
Problem: optimal choice of t

Oracle: $t^\star = \max_t \text{SNR}(x^0, x^{(t)})$.



$\{x_i\}_i$

$\{x_i^{(t^\star)}\}_i$



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$$\tilde{L} = (G D^{-1/2})^{\mathrm{T}} (G D^{-1/2}) \quad \Longrightarrow \quad \tilde{L} \text{ is positive semi-definite.}$$

Eigen-decomposition of the Laplacian: $\exists U, \quad U^{\mathrm{T}} U = \mathrm{Id}_n,$

$$\tilde{L} = U \Lambda U^{\mathrm{T}} \qquad \text{where} \qquad \Lambda = \mathrm{diag}_{\omega}(\lambda_{\omega}) \qquad \text{and} \qquad \lambda_1 \leq \ldots \leq \lambda_n.$$

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Theorem: $\forall\, i,\ \lambda_i \in [0, 2]$ and

- If M is connected then $0 = \lambda_1 < \lambda_2$.
- $\lambda_n = 2$ if and only if M is 2-colorable.

Laplacian Eigen-decomposition

$$\tilde{L} = D^{-1/2} L D^{-1/2} = \mathrm{Id}_n - D^{-1/2} W D^{-1/2}$$

$$\tilde{L} = (G D^{-1/2})^{\mathrm{T}} (G D^{-1/2}) \quad \Longrightarrow \quad \tilde{L} \text{ is positive semi-definite.}$$

Eigen-decomposition of the Laplacian: $\exists\, U, \quad U^{\mathrm{T}} U = \mathrm{Id}_n$,

$$\tilde{L} = U \Lambda U^{\mathrm{T}} \qquad \text{where} \qquad \Lambda = \mathrm{diag}_\omega(\lambda_\omega) \qquad \text{and} \qquad \lambda_1 \leq \ldots \leq \lambda_n.$$

Theorem: $\forall\, i,\ \lambda_i \in [0, 2]$ and

- If M is connected then $0 = \lambda_1 < \lambda_2$.
- $\lambda_n = 2$ if and only if M is 2-colorable.

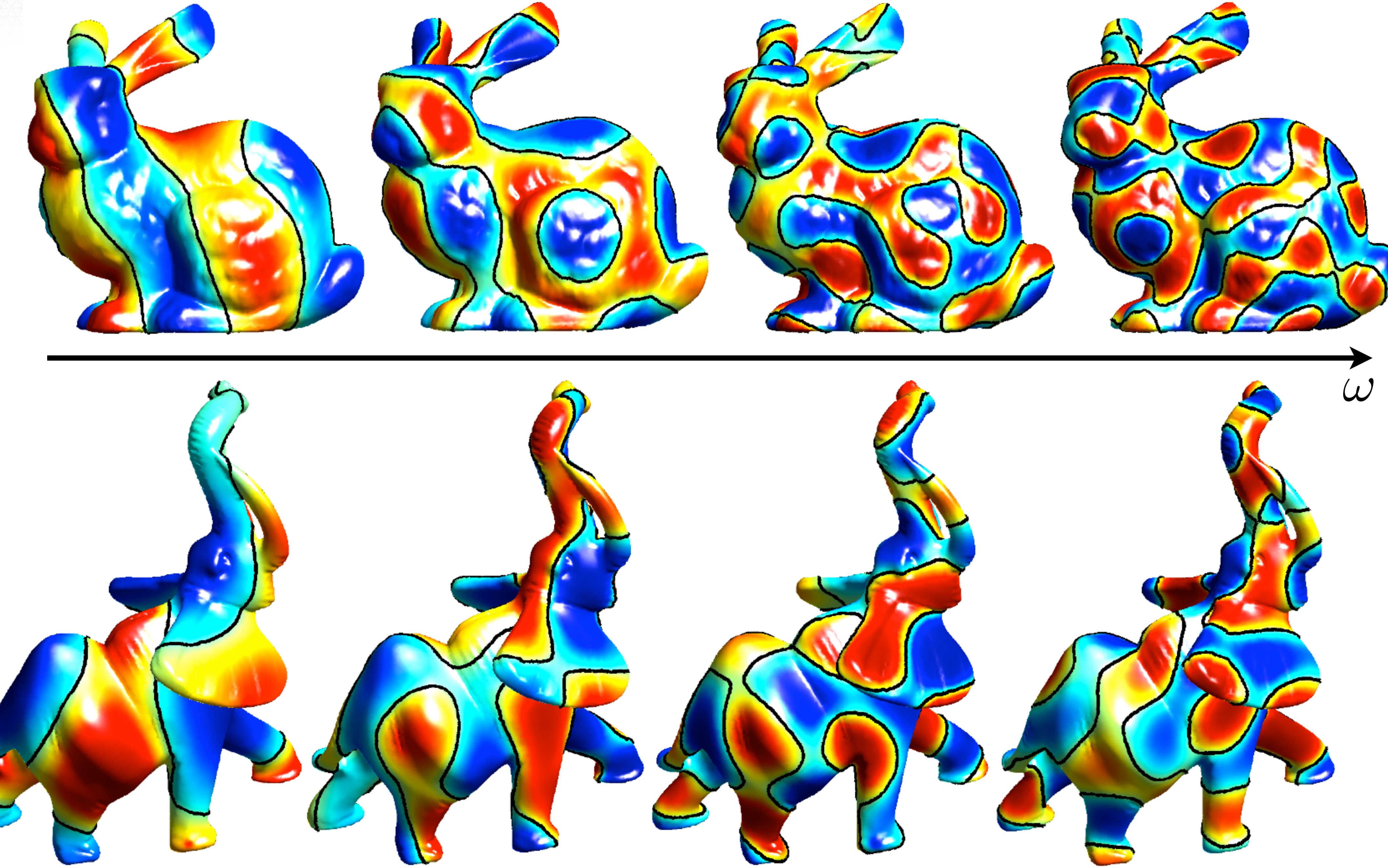
Eigen-basis: $U = (u_\omega)_\omega$ orthogonal basis of $\mathbb{R}^n \simeq \ell^2(V)$.

$$u_\omega : \begin{cases} V & \longrightarrow & \mathbb{R} \\ i & \longmapsto & u_\omega(i) \end{cases}$$

Orthogonal expansion: $\langle u_\omega,\, u_{\omega'} \rangle = \delta_\omega^{\omega'}$,

$$\forall\, f \in \ell^2(V), \quad f = \sum_\omega \langle f,\, u_\omega \rangle u_\omega.$$

Eigenvectors of the Laplacian



Examples of Eigen-decompositions

Laplacian in 1-D: $Lf = (-1/2, 1, -1/2) \star f$

Theorem (in 1D):

$$u_\omega(k) = n^{-1/2} e^{\frac{2i\pi}{n} k\omega} \qquad \lambda_\omega = \sin^2\left(\frac{\pi}{n}\omega\right)$$

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Laplacian in 2-D: $$Lf = \begin{pmatrix} 0 & -1/4 & 0 \\ -1/4 & 1 & -1/4 \\ 0 & -1/4 & 0 \end{pmatrix} \star f$$

Theorem (in 2D): $n = n_1 n_2$, $\omega = (\omega_1, \omega_2)$

$$u_\omega(k) = n^{-1} e^{\frac{2i\pi}{n} \langle k, \omega \rangle} \qquad \lambda_\omega = \frac{1}{2}\sin^2\left(\frac{\pi}{n}\omega_1\right) + \frac{1}{2}\sin^2\left(\frac{\pi}{n}\omega_2\right)$$

On a 3D mesh: $(u_\omega)_\omega$ is the extension of the Fourier basis.

Fourier Transform on Meshes

Manifold-Fourier transform: for $f \in \ell^2(V)$,

$$\Phi(f)(\omega) = \hat{f}(\omega) \overset{\text{def.}}{=} \langle D^{1/2} f,\, u_\omega \rangle \quad \Longleftrightarrow \quad \begin{cases} \Phi(f) = \hat{f} = U^{\mathrm{T}} D^{1/2} f, \\ \Phi^{-1}(\hat{f}) = D^{-1/2} U \hat{f}. \end{cases}$$

Theorem: $\Phi \tilde{W} \Phi^{-1} = \mathrm{Id}_n - \Lambda,$

$$\Longrightarrow \quad \widehat{\tilde{W} f}(\omega) = (1 - \lambda_\omega) \hat{f}(\omega)$$

Proof: $$\Phi \tilde{W} \Phi^{-1} = U^* \underbrace{D^{1/2} W D^{-1/2}}_{\mathrm{Id}_n - \tilde{L}} U = \mathrm{Id}_n - \Lambda$$

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Theorem: if $\lambda_n < 2$ (i.e. M is not 2-colorable),

$$\tilde{W}^k f \overset{k \to +\infty}{\longrightarrow} f^\star = \langle f,\, d \rangle d^{-1}$$

$$\tilde{W}^k f = \Phi^{-1} (\mathrm{Id}_n - \Lambda)^k \Phi(f) \longrightarrow f^\star$$

Mesh Approximation and Compression

Orthogonal basis $U = (u_\omega)_\omega$ of $\ell^2(V) \simeq \mathbb{R}^n$, where $\tilde{L} = U \Lambda U^{\mathrm{T}}$.

$$f = \sum_{\omega=1}^{n} \langle f,\, u_\omega \rangle u_\omega \quad \overset{M\text{-term approx.}}{\Longrightarrow} \quad f_M \overset{\text{def.}}{=} \sum_{\omega=1}^{M} \langle f,\, u_\omega \rangle u_\omega.$$

Error decay: $E(M) \overset{\text{def.}}{=} \|f - f_M\|^2 = \sum_{\omega > M} |\langle f,\, u_\omega \rangle|^2$.

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Good basis $\iff$ $E(M)$ decays fast.

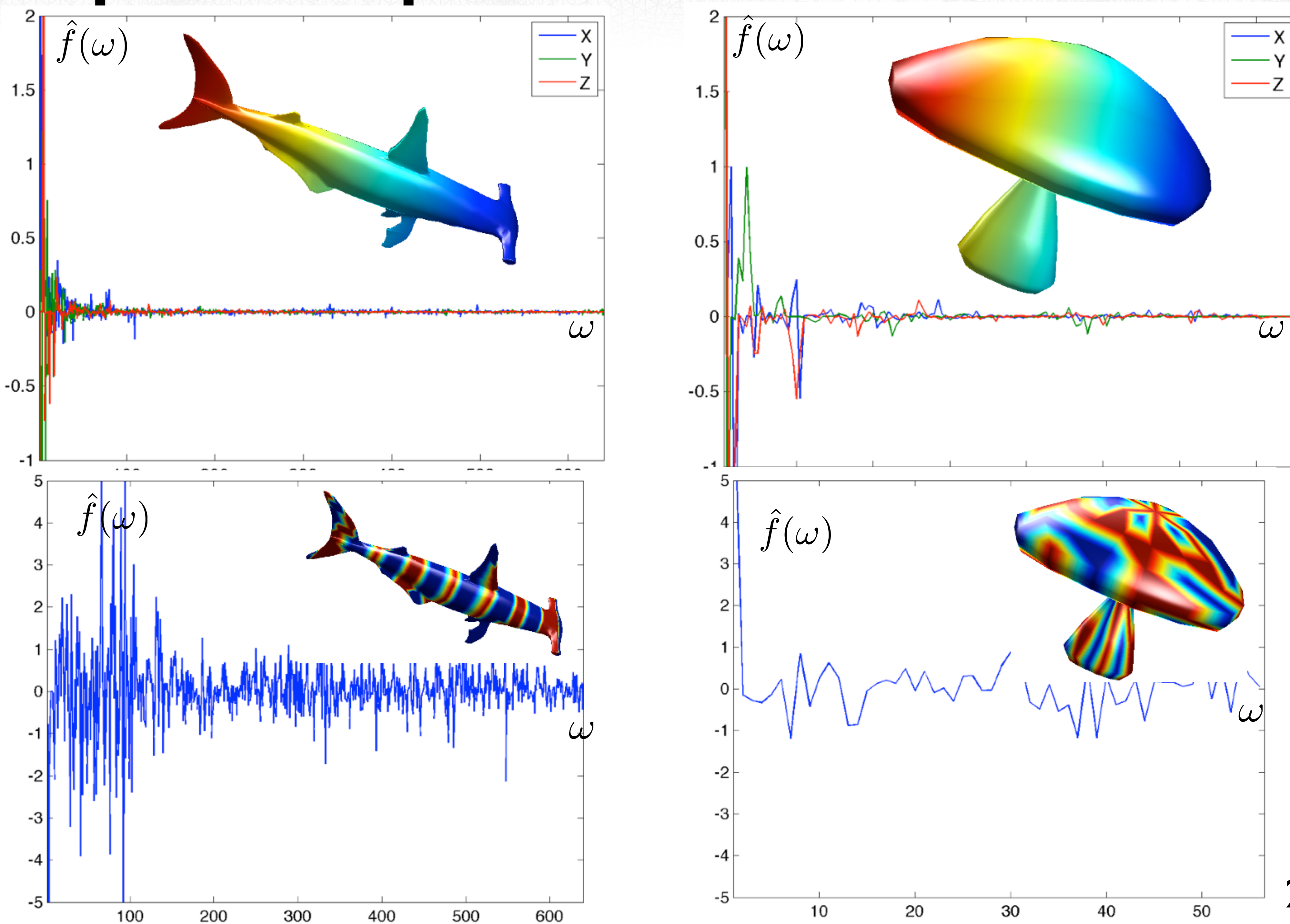
Example in 1D: if f is C^α on $\mathbb{R}/(2\pi\mathbb{Z})$, $\quad |\hat{f}(\omega)| \leq \|f^{(\alpha)}\|_\infty |\omega|^{-\alpha}.$

Example on a mesh: f is smooth if $\|\tilde{L} f\|$ is small.

$$|\langle f,\, u_\omega \rangle| = \frac{1}{\lambda_\omega} |\langle f,\, \tilde{L} u_\omega \rangle| \leqslant \frac{1}{\lambda_\omega} \|\tilde{L} f\|$$

Intuition: $\lambda_\omega \sim |\omega|^2$.

Laplace Spectrum



Mesh Compression