Mathematical introduction to Compressed Sensing

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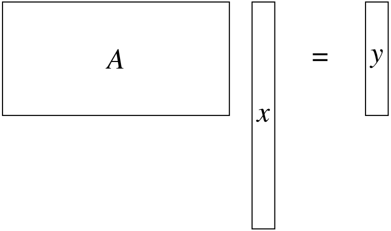
Problem in Compressed Sensing

Find x such that $y = Ax$ from (y, A) (when $m << n$) knowing that x is sparse



CS = solve a highly undetermined linear system under sparsity assumption

Compressed sensing: problems statement

Problem 1: *minimal number of measurements and construction of compression matrix A:* *construct $A \in \mathbb{R}^{m \times n}$ such that one can reconstruct all s-sparse signal x from the m measurements $y = Ax$ with a minimal number of measurements m.*

Problem 2: *Construct efficient algorithms* *that can reconstruct exactly any s-sparse signal x from the measurements $y = Ax$.*

A necessary condition

Definition

We say that $A \in \mathbb{R}^{m\times n}$ satisfies the **(CN(s))** when

$$\forall u, v \in \Sigma_s = \{x \in \mathbb{R}^n : \|x\|_0 \leq s\}, \quad Au \neq Av$$

Theorem

*If $A \in \mathbb{R}^{m\times n}$ satisfies **(CN(s))** then $m \geq 2s$.*

Theorem

The following are equivalent

1. *A satisfies **(CN(s))***
2. $\mathrm{Ker}(A) \cap \Sigma_{2s} = \{0\}$ *(and so $m \geq 2s$)*
3. *all $m \times 2s$ sub-matrix A are one-to-one (injective).*

Properties of the ℓ_0-minimization procedure

the ℓ_0-minimization procedure

Look for the sparsest solution of the system $y = Ax$:

$$\boxed{\hat{x}_0 \in \underset{At=y}{\operatorname{argmin}} \|t\|_0}$$

Definition

$\hat{x}_0$ is called the ℓ_0-minimization procedure

Definition

We say that $A \in \mathbb{R}^{m \times n}$ satisfies $(P_{\ell_0,s})$ property when

$$\forall x \in \Sigma_s, \quad \underset{At=Ax}{\operatorname{argmin}} \|t\|_0 = \{x\}. \tag{1}$$

Theorem

The following are equivalent

1. *A satisfies* $(P_{\ell_0,s})$
2. *A satisfies* ***(CN(s))***

Theorem

For all $n \geq 2s$, there exists $A \in \mathbb{R}^{m \times n}$ such that :

1. $m = 2s$
2. *A satisfies* ***(CN(s))***

Vandermonde matrix: let $t_N > \cdots > t_1 > 0$ and define

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_N \\ \vdots & \vdots & \vdots & \vdots \\ t_1^{2s-1} & t_2^{2s-1} & \cdots & t_N^{2s-1} \end{pmatrix}.$$

But we cannot use the ℓ_0-minimization procedure in practice.

Properties of the ℓ_1-minimization procedure

Definition

We say that $A \in \mathbb{R}^{m\times n}$ satisfies $(P_{\ell_1,s})$ property when

$$\forall x \in \Sigma_s, \quad \underset{At=Ax}{\operatorname{argmin}} \|t\|_1 = \{x\}. \tag{2}$$

Theorem

If $A \in \mathbb{R}^{m\times n}$ satisfies $(P_{\ell_1,s})$ then necessarily

$$m \geq c_0 s \log\left(\frac{n}{s}\right)$$

Random matrices

Definition

A <u>Standard Gaussian matrix</u> G is a $m \times n$

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{m1} & \cdots & g_{mn} \end{pmatrix}$$

where $g_{11}, \ldots, g_{mn}$ are mn i.i.d. standard Gaussian variables.

RIP

Definition

Let $A \in \mathbb{R}^{m\times n}$ and $1 \leq s \leq n$. We say that A satisfies the **Restricted Isometry Property of order** s **RIP(**s**)** when for all $x \in \Sigma_s$,

$$\frac{1}{2}\|x\|_2 \leq \frac{\|Ax\|_2}{\sqrt{m}} \leq \frac{3}{2}\|x\|_2 .$$

Theorem

Let $G \in \mathbb{R}^{m\times n}$ *be* $m \times n$ *standard Gaussian matrix. Then with probability at least* $1 - 2\exp(-c_0 m)$, *for all* $u, v \in \Sigma_s$,

$$\frac{1}{2}\|u-v\|_2 \leq \frac{\|Gu-Gv\|_2}{\sqrt{m}} \leq \frac{3}{2}\|u-v\|_2$$

when $m \geq c_1 s \log(en/s)$.

Conclusion

1. Convex relaxation is very efficient
2. Random matrices are very efficient “dimension reduction / compression tools”