An introduction to mean field games and their applications

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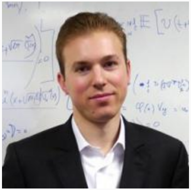
Introduction

The speaker



- PhD thesis on mean field games under the supervision of Pierre-Louis Lions.
- Academic positions at Paris 7, then ENSAE, and now Paris 1.
- Main research field: optimal control and applications (incl. **mean field games**, stochastic optimal control in finance, reinforcement learning, etc.).
- Start-up (MFG Labs) with Lasry and Lions (created in 2009 – acquired in 2013 by Havas).

Mean field games – In the beginning were...

Pierre-Louis Lions and Jean-Michel Lasry, who introduced mean field games (MFG) in 2006.
$\rightarrow$ *Similar ideas arose in electrical engineering (Caines, Huang, Malhamé, 2006)*

Introduction - Mean field games

Game theory

- The study of strategic interactions.
- Central concept of Nash equilibrium.
- In MFG: the number of players is large.

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Mean field

- Approximation as in physics, here to model strategic interactions, not interactions between particles.
- Philosophical difference: freedom... however, as Spinoza said: *This is that human freedom, which all boast that they possess, and which consists solely in the fact, that men are conscious of their own desire, but are ignorant of the causes whereby that desire has been determined.*
- Difference for the maths: humans anticipate, particles do not! **Main consequence: the equations are not simply forward in time.**

Numerous applications

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Economics

- Economic growth and inequality.
- Oil extraction.
- Mining industries.
- Labor market.
- etc.

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Population dynamics

- Waves in stadiums (ola).
- Structure of cities.
- Traffic jam and other forms of congestion.
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Finance

- Competition between asset managers.
- Optimal execution of several brokers.
- etc.

Many forms but two main characteristics

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Different forms of mean field games

- Static games / games in discrete time / differential games (continuous time).
- Discrete / continuous state space.

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Two main characteristics

- Continuum of anonymous players.
- All players maximize the same objective function (*possible to generalize to several populations of players*).

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A fixed-point equilibrium approach

- Each infinitesimal player takes the distribution of players as given.
- The distribution of players proceeds from all individual choices.

A large and worldwide community (not exhaustive and slightly outdated)

- Collège de France: P.-L. Lions + J.-M. Lasry
- Dauphine: P. Cardaliaguet, J. Salomon, G. Turinici
- Paris-Diderot: Y. Achdou + Ph.D. students
- Nice: F. Delarue
- Italy (Roma + Padua): I. Capuzzo-Dolcetta, F. Camilli, M. Bardi
- Princeton: R. Carmona (+ Ph.D. students), B. Moll (econ)
- Columbia: Daniel Lacker
- Chicago: R. Lucas (econ)
- McGill: P. Caines + collaborators around the world
- KAUST: D. Gomes (+ Ph.D. students), P. Markowich
- Hong Kong + Dallas: A. Bensoussan

Some references

Some references

Initial papers

- J.-M. Lasry and P.-L. Lions. Jeux à champ moyen i. le cas stationnaire. C. R. Acad. Sci. Paris, 343(9), 2006.
- J.-M. Lasry and P.-L. Lions. Jeux à champ moyen ii. horizon fini et contrôle optimal. C. R. Acad. Sci. Paris, 343(10), 2006.
- J.-M. Lasry and P.-L. Lions. Mean field games. Japanese Journal of Mathematics, 2(1), Mar. 2007.

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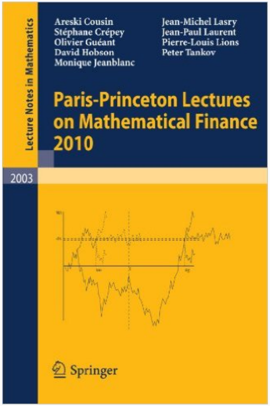
Courses and notes

- 5 years of PLL's lectures about MFG available on the website of the Collège de France (in French).
- Notes by P. Cardaliaguet

Some references (cont'd)

Some applications: O. Guéant, J.-M. Lasry and P.-L. Lions. Mean field games and applications, *in* Paris-Princeton Lectures on Mathematical Finance, 2010



Static mean field games

Reminder about game theory

- Game theory studies strategic interactions.
- N players. Strategies $(x_1, x_2, \dots, x_N) \in E^N$ (E compact set).
- Player i has utility (or score) $u_i(x_i, x_{-i})$.

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- Player i has utility (or score) $u_i(x_i, x_{-i})$.
- Key notion: Nash equilibrium

Nash equilibrium

$(x_1^*, \dots, x_N^*)$ is a Nash equilibrium $\iff$ for any player i, x_i^* is the best strategy when others play x_{-i}^*.

i.e.:

$$\forall i, \quad x_i^* \text{ maximizes } x_i \mapsto u_i(x_i, x_{-i}^*).$$

$N \to +\infty$

Mean field hypotheses

- Players have the same objective function $u_i = u$.
- Players are anonymous: $\forall x_i, \quad x_{-i} \mapsto u(x_i, x_{-i})$ is a symmetrical function.

$$u(x_i, x_{-i}) = u\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) = u\left(x_i, m^{N-1}(x_{-i})\right).$$

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Static MFGs

A static MFG is given by a function

$$U : (x, m) \in E \times \mathcal{P}(E) \mapsto U(x, m),$$

where m stands for the distribution of the players' strategies.
Remark: $\mathcal{P}(E)$ is the (compact) set of probability measures on E.

Nash-MFG equilibrium

What is a Nash equilibrium when N tends to $+\infty$?

- A Nash equilibrium with N players is a tuple $(x_1^*, x_2^*, \dots, x_N^*)$. When $N \to +\infty$, an equilibrium is a probability measure m.

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Definition: Nash-MFG

m is a Nash-MFG equilibrium

$\iff$ The support of m is included in the argmax of $x \mapsto U(x, m)$

$\iff$ For any probability measure $f \in \mathcal{P}(E)$ on the set of strategies E,

$$\int_E U(x, m)m \geq \int_E U(x, m)f$$

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This definition shows that m solves a (rather uncommon) fixed-point problem.

Underlying mathematical result

Theorem

- Let us assume that U is continuous.
- Let us consider a sequence $((x_1^N, \dots, x_N^N))_N$ where $\forall N, (x_1^N, \dots, x_N^N)$ is a Nash equilibrium of the N-player game corresponding to $\mathcal{U}_{|E \times E^N/\mathfrak{S}^N}$.

Then, up to a subsequence, $\exists m \in \mathcal{P}(E)$ such that:

1. $m^N(x_1^N, \dots, x_N^N)$ weakly converges towards m.
2. m is a Nash-MFG equilibrium.

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Remark: this results can also be adapted to prove an existence result (by using mixed strategies).

What about uniqueness?

Uniqueness

If U is decreasing in the sense that

$$\forall m_1 \neq m_2, \quad \int (U(x, m_1) - U(x, m_2))(m_1 - m_2) < 0$$

then, an equilibrium is unique.

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This type of monotonicity result is ubiquitous in the MFG literature.

MFG and planning

Variational characterization (planner's problem)

If there exists a function $m \mapsto F(m)$ on $\mathcal{P}(E)$ such that $DF = U$, then any maximum of F is a Nash-MFG equilibrium.

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- Objective function: $U(x, m) = -x^2 - \gamma m(x)$.

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- Global problem: $F(m) = \int -x^2 m(x) dx - \frac{\gamma}{2} m(x)^2 dx$.
- Unique equilibrium, of the form $m(x) = \frac{1}{\gamma}(\lambda - x^2)_+$

MFG in continuous time (with continuous state space)

Differential games

Static games are interesting but MFGs are really powerful in continuous time (differential games):

The real power of MFGs in continuous time

- Differential/stochastic calculus.
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- Numerical methods.

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Also, very general results have been obtained with probabilistic methods (see Carmona, Delarue).

Reminder of (stochastic) optimal control

Agent's dynamics

$$dX_t = \alpha_t dt + \sigma dW_t, \qquad X_0 = x$$

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Remarks:

- f and L can also include a time dependency (e.g. discount rate).
- Stationary (infinite horizon)/Ergodic problems can also be considered.

Reminder of (stochastic) optimal control

Main tool: value function

The best "score" an agent can expect when he is in x at time t:

$$u(t,x) = \sup_{(\alpha_s)_{s\geq t}} \mathbb{E}\left[\int_t^T (f(X_s) - L(\alpha_s))\,ds + g(X_T) | X_t = x\right]$$

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PDE

u “solves” the Hamilton-Jacobi(-Bellman) equation:

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u) = -f(x), \qquad u(T,x) = g(x),$$

where $H(p) = \sup_\alpha \alpha \cdot p - L(\alpha)$.

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Optimal control

The optimal control is $\alpha^*(t,x) = \nabla H(\nabla u(t,x))$.

From optimal control problems to mean field games

- Continuum of players.

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- Each player has a position X^i that evolves according to:

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Remark: only independent idiosyncratic risks (common noise has also been studied but it is more complicated).

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- Each player optimizes:

$$\max_{(\alpha_s^i)_{s\geq 0}} \mathbb{E}\left[\int_0^T \Big(f(X_s^i, m(s,\cdot)) - L(\alpha_s^i, m(s,\cdot))\Big)\, ds + g(X_T^i, m(T,\cdot))\right]$$

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- The Nash-equilibrium $t \in [0,T] \mapsto m(t,\cdot)$ must be consistent with the decisions of the agents.

Examples

Repulsion

- $f(x, m) = -m(t, x) - \delta x^2$ and $g = 0$.
  $\rightarrow$ *Willingness to be close to* 0 *but far from other players.*
- Quadratic cost: $L(\alpha) = \frac{\alpha^2}{2}$.

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Congestion

Cost of the form $L(\alpha, m(t, x)) = \frac{\alpha^2}{2}(1 + m(t, x))$.

Partial differential equations

- u value function of the control problem (with given m).
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MFG PDEs

$$(HJB) \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u, m) = -f(x, m)$$

$$(K) \qquad \partial_t m + \nabla \cdot (m \nabla_p H(\nabla u, m)) = \frac{\sigma^2}{2}\Delta m$$

where $H(p, m) = \sup_\alpha \alpha \cdot p - L(\alpha, m)$.

$$u(T, x) = g(x), \qquad m(0, x) = m_0(x)$$

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The optimal control is $\alpha^*(t, x) = \nabla_p H(\nabla u(t, x), m(t, \cdot))$.

Remarks and variants

Forward/Backward

The system of PDEs is a forward/backward problem:

- The HJB equation is backward in time (terminal condition) because agents anticipate the future.
- The transport equation is forward in time because it corresponds to the dynamics of the agents.

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Related problem

Same equations with initial and final conditions on m and no terminal condition on u: the problem is then that of finding the right terminal payoff g so that agents go from m_0 to m_T.

Some results

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Existence

A wide variety of PDE results, depending on f, L, g and σ.

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Uniqueness

If the cost function L **does not depend** on m and if f is decreasing in the sense:

$$\forall m_1 \neq m_2, \quad \int (f(x, m_1) - f(x, m_2))(m_1 - m_2) < 0$$

then a solution of the PDEs system is unique.

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Remarks:

- Same criterion as above.
- For more general cost functions L (*e.g.* congestion), there is a more general criterion (see Lions, or see the result in graphs).

MFG with quadratic cost/Hamiltonian

MFG equations with quadratic cost function $L(\alpha) = \frac{\alpha^2}{2}$ on the domain $[0, T] \times \Omega$, Ω standing for $(0,1)^d$:

$$\text{(HJB)} \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = -f(x, m)$$

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Examples of conditions

- Boundary conditions: $\frac{\partial u}{\partial n} = \frac{\partial m}{\partial n} = 0$ on $(0, T) \times \partial\Omega$
- Terminal condition: $u(T, x) = g(x)$.
- Initial condition: $m(0, x) = m_0(x) \geq 0$.

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The optimal control is $\alpha^*(t,x) = \nabla u(t,x)$.

Change of variables

Theorem: $u = \sigma^2 \log(\phi)$, $m = \phi\psi$

Let's consider a smooth solution (ϕ, ψ) (with $\phi > 0$) of:

$$\begin{aligned} \partial_t \phi + \frac{\sigma^2}{2}\Delta\phi &= -\frac{1}{\sigma^2} f(x, \phi\psi)\phi \qquad (E_\phi) \\ \partial_t \psi - \frac{\sigma^2}{2}\Delta\psi &= \frac{1}{\sigma^2} f(x, \phi\psi)\psi \qquad (E_\psi) \end{aligned}$$

- Boundary conditions: $\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0$ on $(0, T) \times \partial\Omega$
- Terminal condition: $\phi(T, \cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
- Initial condition: $\psi(0, \cdot) = \frac{m_0(\cdot)}{\phi(0, \cdot)}$

Then $(u, m) = (\sigma^2 \log(\phi), \phi\psi)$ is a solution of (MFG).

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$$\begin{aligned} \partial_t \phi + \frac{\sigma^2}{2}\Delta\phi &= -\frac{1}{\sigma^2} f(x, \phi\psi)\phi \qquad (E_\phi) \\ \partial_t \psi - \frac{\sigma^2}{2}\Delta\psi &= \frac{1}{\sigma^2} f(x, \phi\psi)\psi \qquad (E_\psi) \end{aligned}$$

- Boundary conditions: $\frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial n} = 0$ on $(0, T) \times \partial\Omega$
- Terminal condition: $\phi(T, \cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
- Initial condition: $\psi(0, \cdot) = \frac{m_0(\cdot)}{\phi(0, \cdot)}$

Then $(u, m) = (\sigma^2 \log(\phi), \phi\psi)$ is a solution of (MFG).

Nice existence results exist on this system (see some of my papers).

Numerics and examples

Numerical methods

- Variational formulation: when a global maximization problem exists, gradient-descent/ascent can be used (see Lachapelle, Salomon, Turinici)
- **Finite difference method** (Achdou and Capuzzo-Dolcetta)
- Specific methods in the quadratic cost case (see Guéant).

Examples with population dynamics

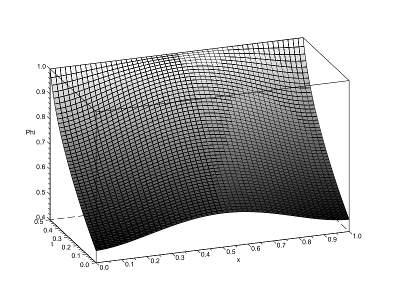
Toy problem in the quadratic case

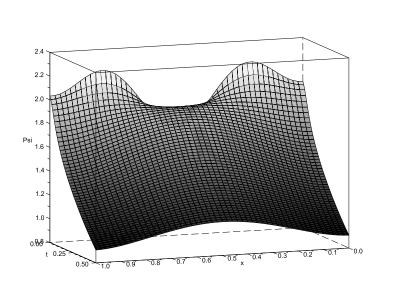
- $f(x,\xi) = -16(x-1/2)^2 - 0.1\max(0,\min(5,\xi))$, *i.e.* agents want to live near $x = \frac{1}{2}$ but they do not want to live together.
- $T = 0.5$
- $g = 0$
- $\sigma = 1$
- $m_0(x) = \frac{\mu(x)}{\int_0^1 \mu(x')dx'}$, where

$$\mu(x) = 1 + 0.2\cos\left(\pi\left(2x - \frac{3}{2}\right)\right)^2.$$

Toy problem in the quadratic case

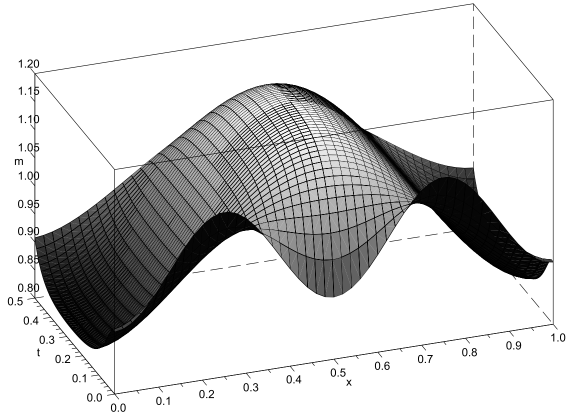
The functions ϕ and ψ.





Toy problem in the quadratic case

The dynamics of the distribution m.



Examples with population dynamics (videos provided by Y. Achdou)

Going out of a movie theater (1)

- We consider a movie theatre with 6 rows, and 2 doors in the front to exit.
- Neumann conditions on walls.
- Homogenous Dirichlet conditions at the doors.
- Running penalty while staying in the room.
- Congestion effects.

Examples with population dynamics (videos provided by Y. Achdou)

Going out of a movie theater (2)

- The same movie theatre with 6 rows, and 2 doors in the front to exit.
- One door only will be open at a pre-defined time, but nobody knows which one.

Numerous economic applications

Many models in economics and finance – for instance:

- Interaction between economic growth and inequalities (where Pareto distributions play a central role).
  $\rightarrow$ See Guéant, Lasry, Lions (Paris-Princeton lectures).
  $\rightarrow$ Similar ideas developed by Lucas and Moll.

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- Oil extraction (*à la* Hotelling) with noise.
  → See Guéant, Lasry, Lions (Paris-Princeton lectures).
- A long-term model for the mining industries.
  → Joint work of Achdou, Giraud, Lasry, Lions, and Scheinkman.

Special for Huawei: MFG on graphs

Framework

MFGs are often written on continuous state spaces, but what about discrete structures?

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Notations for graph

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Notations for graph

- Graph $\mathcal{G}$. Nodes indexed by integers from 1 to N.

Framework

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Notations for graph

- Graph $\mathcal{G}$. Nodes indexed by integers from 1 to N.
- $\forall i \in \mathcal{N} = \{1, \ldots, N\}$:
  - $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from i to j (cardinal: d_i).
  - $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from j to i.

Framework (continued)

Players, strategies, and costs

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- Each player's position: Markov chain $(X_t)_t$ with values in $\mathcal{G}$.

Framework (continued)

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- Each player's position: Markov chain $(X_t)_t$ with values in $\mathcal{G}$.
- Instantaneous transition probabilities at time t:
  $\lambda_t(i, \cdot) : \mathcal{V}(i) \to \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$

Framework (continued)

Players, strategies, and costs

- Each player's position: Markov chain $(X_t)_t$ with values in $\mathcal{G}$.
- Instantaneous transition probabilities at time t:
  $\lambda_t(i,\cdot) : \mathcal{V}(i) \to \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$
- Instantaneous cost $L(i,(\lambda_{i,j})_{j\in\mathcal{V}(i)})$ to set the value of $\lambda(i,j)$ to $\lambda_{i,j}$.

Hypotheses

Hypotheses on L

- Super-linearity hypothesis:

$$\forall i \in \mathcal{N}, \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \to +\infty} \frac{L(i, \lambda)}{|\lambda|} = +\infty$$

- Convexity hypothesis:
  $\forall i \in \mathcal{N}, \lambda \in \mathbb{R}_+^{d_i} \mapsto L(i, \lambda)$ is strictly convex.

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  $\forall i \in \mathcal{N}, \lambda \in \mathbb{R}_+^{d_i} \mapsto L(i,\lambda)$ is strictly convex.

Also, we define:

$$\forall i \in \mathcal{N}, p \in \mathbb{R}^{d_i} \mapsto H(i,p) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - L(i,\lambda).$$

Mean field game - control problem

Control problem

Mean field game - control problem

Control problem

- Admissible Markovian controls:

$$\mathcal{A} = \left\{ (\lambda_t(i,j))_{t\in[0,T], i\in\mathcal{N}, j\in\mathcal{V}(i)} \,\middle|\, t \mapsto \lambda_t(i,j) \in L^\infty(0,T) \right\}$$

Mean field game - control problem

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- For $\lambda \in \mathcal{A}$ and a given function $m : [0,T] \mapsto \mathcal{P}_N$ we define the payoff function: $J_m : [0,T] \times \mathcal{N} \times \mathcal{A} \to \mathbb{R}$ by:

$$J_m(t,i,\lambda) = \mathbb{E}\Bigg[\int_t^T \left(-L(X_s, \lambda_s(X_s,\cdot)) + f(X_s, m(s))\right) ds + g\left(X_T, m(T)\right) \Bigg]$$

for $(X_s)_{s \in [t,T]}$ a Markov chain on $\mathcal{G}$, starting from i at time t, with instantaneous transition probabilities given by $(\lambda_s)_{s \in [t,T]}$.

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Nash equilibrium

Nash-MFG equilibrium

A differentiable function $m : t \in [0, T] \mapsto (m(t,i))_i \in \mathcal{P}_N$ is said to be a Nash-MFG equilibrium, if there exists an admissible control $\lambda \in \mathcal{A}$ such that:

$$\forall \tilde{\lambda} \in \mathcal{A}, \forall i \in \mathcal{N}, J_m(0, i, \lambda) \geq J_m(0, i, \tilde{\lambda})$$

and

$$\forall i \in \mathcal{N}, \frac{d}{dt} m(t,i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_t(j,i) m(t,j) - \sum_{j \in \mathcal{V}(i)} \lambda_t(i,j) m(t,i)$$

In that case, λ is called an optimal control.

The $\mathcal{G}$-MFG equations

Definition (The $\mathcal{G}$-MFG equations)

The $\mathcal{G}$-MFG equations consist in a system of $2N$ equations, the unknown being $t \in [0, T] \mapsto (u(t), m(t))$:

$$\forall i \in \mathcal{N}, \quad \frac{d}{dt}u(t,i) + H\left(i, (u(t,j) - u(t,i))_{j\in\mathcal{V}(i)}\right) + f(i, m(t)) = 0,$$

$$\forall i, \frac{d}{dt}m(t,i) = \sum_{j\in\mathcal{V}^{-1}(i)} \frac{\partial H(j,\cdot)}{\partial p_i}\left((u(t,k) - u(t,j))_{k\in\mathcal{V}(j)}\right) m(t,j)$$

$$- \sum_{j\in\mathcal{V}(i)} \frac{\partial H(i,\cdot)}{\partial p_j}\left((u(t,k) - u(t,i))_{k\in\mathcal{V}(i)}\right) m(t,i)$$

with $u(T,i) = g(i, m(T))$ and $m(0) = m^0 \in \mathcal{P}_N$ given.

The $\mathcal{G}$-MFG equations

Proposition (The $\mathcal{G}$-MFG equations as a sufficient condition)

Let $m^0 \in \mathcal{P}_N$ and let us consider a C^1 solution $(u(t), m(t))$ of the $\mathcal{G}$-MFG equations with $(m(0,1), \ldots, m(0,N)) = m^0$.

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Then:

- $t \mapsto m(t) = (m(t,1), \dots, m(t,N))$ *is a Nash-MFG equilibrium*
- *The relations* $\lambda_t(i,j) = \frac{\partial H(i,\cdot)}{\partial p_j}\left((u(t,k) - u(t,i))_{k \in \mathcal{V}(i)}\right)$ *define an optimal control.*

Existence of a solution

Existence of a solution

Proposition (Existence of a solution to the $\mathcal{G}$-MFG equations)

Let $m^0 \in \mathcal{P}_N$. Under the assumptions made above, there exists a C^1 solution (u, m) of the $\mathcal{G}$-MFG equations such that $m(0) = m^0$.

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Sketch of proof (Fixed point):

- Comparison principle leads a priori bounds on u

$$\sup_{i \in \mathcal{N}} \|u(\cdot, i)\|_\infty \leq \sup_{i \in \mathcal{N}} \|g(i, \cdot)\|_\infty + \left(\sup_{i \in \mathcal{N}} \|f(i, \cdot)\|_\infty + \sup_{i \in \mathcal{N}} |H(i, 0)| \right) T.$$

- $\Rightarrow$ bounds on $\frac{dm}{dt}$.
- Ascoli + Schauder to conclude.

Uniqueness of smooth solutions

Proposition (Uniqueness for the solution of the $\mathcal{G}$-MFG equations)

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Assume that f and g are such that:

$$\forall (m,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (f(i,m) - f(i,\mu))(m_i - \mu_i) \geq 0 \implies m = \mu$$

and

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$$\forall (m,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (g(i,m) - g(i,\mu))(m_i - \mu_i) \geq 0 \implies m = \mu$$

Then, if $(\widehat{u}, \widehat{m})$ and $(\tilde{u}, \tilde{m})$ are two C^1 solutions of the $\mathcal{G}$-MFG equations, we have $\widehat{m} = \tilde{m}$ and $\widehat{u} = \tilde{u}$.

The $\mathcal{G}$-Master equations

Definition (The $\mathcal{G}$-Master equations)

The $\mathcal{G}$-Master equations consist in N equations, the unknown being $(t, m) \in [0, T] \times \mathcal{P}_N \mapsto (U_1(t, m), \ldots, U_N(t, m))$.

$$\forall i \in \mathcal{N}, \quad \frac{\partial U_i}{\partial t}(t, m) + H\left(i, (U_j(t, m) - U_i(t, m))_{j \in \mathcal{V}(i)}\right)$$

$$+ \sum_{l=1}^{N} \frac{\partial U_i}{\partial m_l}(t, m) \left[\sum_{j \in \mathcal{V}^{-1}(l)} \frac{\partial H(j, \cdot)}{\partial p_l} \left((U_k(t, m) - U_j(t, m))_{k \in \mathcal{V}(j)} \right) m_j \right.$$

$$\left. - \sum_{j \in \mathcal{V}(l)} \frac{\partial H(l, \cdot)}{\partial p_j} \left((U_k(t, m) - U_l(t, m))_{k \in \mathcal{V}(l)} \right) m_l \right] + f(i, m) = 0$$

with $U_i(T, m) = g(i, m)$.

The $\mathcal{G}$-Master equations

Proposition (From $\mathcal{G}$-Master equations to $\mathcal{G}$-MFG equations)

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If $(t, m) \in [0, T] \times \mathcal{P}_N \mapsto (U_1(t, m), \ldots, U_N(t, m))$ *is a* C^1 *solution to the* $\mathcal{G}$*-Master equations.*

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If a function m *is such that* $m(0) = m^0 \in \mathcal{P}_N$ *and* $\frac{d}{dt} m(t, i) =$

$$\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial H(j, \cdot)}{\partial p_i} \left((U_k(t, m(t)) - U_j(t, m(t)))_{k \in \mathcal{V}(j)} \right) m(t, j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i, \cdot)}{\partial p_j} \left((U_k(t, m(t)) - U_i(t, m(t)))_{k \in \mathcal{V}(i)} \right) m(t, i)$$

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$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i,\cdot)}{\partial p_j} \left((U_k(t,m(t)) - U_i(t,m(t)))_{k \in \mathcal{V}(i)} \right) m(t,i)$$

Then $t \in [0,T] \mapsto (U_1(t,m(t)), \dots, U_N(t,m(t)), m(t))$ *is a solution of the* $\mathcal{G}$*-MFG equations.*

Potential games

Assumptions

We suppose that there exist two C^1 functions:

$$F : (m_1, \ldots, m_N) \in \mathcal{P}_N \mapsto F(m_1, \ldots, m_N)$$

$$G : (m_1, \ldots, m_N) \in \mathcal{P}_N \mapsto G(m_1, \ldots, m_N)$$

such that $\forall i \in \mathcal{N}$:

$$\frac{\partial F}{\partial m_i} = f(i, \cdot)$$

$$\frac{\partial G}{\partial m_i} = g(i, \cdot)$$

Planning problem

We introduce for $t \in [0, T]$, $m^t \in \mathcal{P}_N$ and a given admissible control (function) $\lambda \in \mathcal{A}$, the payoff function

$$\mathcal{J}(t, m^t, \lambda) =$$
$$\int_t^T \left(F(m(s)) - \sum_{i=1}^{N} L(i, (\lambda_s(i,j))_{j \in \mathcal{V}(i)}) m(s,i) \right) ds + G(m(T))$$

where $\forall i \in \mathcal{N}, m(t,i) = m_i^t$ and $\forall i \in \mathcal{N}, \forall s \in [t, T]$:

$$\frac{d}{ds} m(s,i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j,i) m(s,j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i,j) m(s,i)$$

Optimization problem

The (deterministic) optimization problem we consider is, for a given $m^0 \in \mathcal{P}_N$:

$$\sup_{\lambda \in \mathcal{A}} \mathcal{J}(0, m^0, \lambda).$$

HJ equation

Definition (The $\mathcal{G}$-planning equation)

The $\mathcal{G}$-planning equation consists in one PDE in $\Phi(t,m)$:

$$\frac{\partial \Phi}{\partial t}(t, m_1, \ldots, m_N) + \mathcal{H}(m_1, \ldots, m_N, \nabla\Phi) + F(m_1, \ldots, m_N) = 0$$

with the terminal conditions $\Phi(T,m) = G(m)$, where:

$$\mathcal{H}(m,p) = \sup_{(\lambda_{i,j})_{i\in\mathcal{N}, j\in\mathcal{V}(i)}} \sum_{i=1}^{N} \left[\left(\sum_{j\in\mathcal{V}^{-1}(i)} \lambda_{j,i} m_j - \sum_{j\in\mathcal{V}(i)} \lambda_{i,j} m_i \right) p_i \right.$$

$$\left. - L(i, (\lambda_{i,j})_{j\in\mathcal{V}(i)}) m_i \right] = \sum_{i=1}^{N} m_i H\left(i, (p_j - p_i)_{j\in\mathcal{V}(i)}\right)$$

Solving the planning problem

Proposition

Let us consider a C^1 function Φ solution of the $\mathcal{G}$-planning equation. Then, Φ restricted to $[0, T] \times \mathcal{P}_N$ is the value function of the above planning problem, i.e.:

$$\forall (t, m^t) \in [0, T] \times \mathcal{P}_N, \Phi(t, m^t) = \sup_{\lambda \in \mathcal{A}} \mathcal{J}(t, m^t, \lambda)$$

Proposition

Moreover, if we define $\forall i \in \mathcal{N}, m(t,i) = m_i^t$ *and* $\forall i \in \mathcal{N}, \forall s \in [t,T]$

$$\frac{d}{ds} m(s,i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j,i) m(s,j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i,j) m(s,i)$$

with

$$\lambda_s(i,j) = \frac{\partial H(i,\cdot)}{\partial p_j} \left(\left(\frac{\partial \Phi}{\partial m_k}(s, m(s)) - \frac{\partial \Phi}{\partial m_i}(s, m(s)) \right)_{k \in \mathcal{V}(i)} \right)$$

then λ *is an optimal control for the planning problem.*

Going back to MFG

Proposition

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Let Φ be a C^2 solution of the $\mathcal{G}$-planning equation.

Going back to MFG

Proposition

Let Φ *be a* C^2 *solution of the* $\mathcal{G}$*-planning equation.*
Define $\forall i \in \mathcal{N}$, $\forall t \in [0, T]$, $\forall m \in \mathcal{P}_N$, $U_i(t, m) = \frac{\partial \Phi}{\partial m_i}(t, m)$.

Going back to MFG

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Define $\forall i \in \mathcal{N}$, $\forall t \in [0, T]$, $\forall m \in \mathcal{P}_N$, $U_i(t, m) = \frac{\partial \Phi}{\partial m_i}(t, m)$.
Then, $\nabla \Phi = U = (U_1, \dots, U_N)$ *verifies the $\mathcal{G}$-Master equations.*

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Proposition

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Define $\forall i \in \mathcal{N}$, $\forall t \in [0, T]$, $\forall m \in \mathcal{P}_N$, $U_i(t, m) = \frac{\partial \Phi}{\partial m_i}(t, m)$.
Then, $\nabla \Phi = U = (U_1, \ldots, U_N)$ *verifies the $\mathcal{G}$-Master equations.*

Consequently, if we define $\forall i \in \mathcal{N}, m(0, i) = m_i^0$ *for a given* $m^0 \in \mathcal{P}_N$ *and* $\forall i \in \mathcal{N}, \forall s \in [t, T]$:

$$\frac{d}{ds} m(s, i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j, i) m(s, j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i, j) m(s, i)$$

with $\lambda_s(i, j) = \frac{\partial H(i, \cdot)}{\partial p_j} \left(\left(\frac{\partial \Phi}{\partial m_k}(s, m(s)) - \frac{\partial \Phi}{\partial m_i}(s, m(s)) \right)_{k \in \mathcal{V}(i)} \right)$

then m is a Nash-MFG equilibrium and λ is an optimal control for the initial mean field game (the decentralized problem).

Extending to models with congestion

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Assumptions

- Continuity: $\forall i \in \mathcal{N}, \mathcal{L}(i, \cdot, \cdot)$ is a continuous function from $\mathbb{R}_+^{d_i} \times \mathcal{P}_N$ to $\mathbb{R}$
- Asymptotic super-linearity:
$\forall i \in \mathcal{N}, \forall m \in \mathcal{P}_N, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \to +\infty} \frac{\mathcal{L}(i, \lambda, m)}{|\lambda|} = +\infty$

Extending to models with congestion

Hamiltonian functions:

$$\forall i \in \mathcal{N}, p \in \mathbb{R}^{d_i}, m \in \mathcal{P}_N \mapsto H(i,p,m) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i,\lambda,m)$$

Extending to models with congestion

Hamiltonian functions:

$$\forall i \in \mathcal{N}, p \in \mathbb{R}^{d_i}, m \in \mathcal{P}_N \mapsto H(i,p,m) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i,\lambda,m)$$

Hypotheses

- $\forall i \in \mathcal{N}$, $H(i,\cdot,\cdot)$ is a continuous function.
- $\forall i \in \mathcal{N}, \forall m \in \mathcal{P}_N$, $H(i,\cdot,m)$ is a C^1 function with:

$$\frac{\partial H}{\partial p}(i,p,m) = \mathrm{argmax}_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i,\lambda,m)$$

- $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i)$, $\frac{\partial H}{\partial p_j}(i,\cdot,\cdot)$ is a continuous function.

Extending to models with congestion - Existence

Using the same proof as above:

Proposition (Existence)

Under the assumptions made above, there exists a C^1 solution (u, m) of the $\mathcal{G}$-MFG equations.

Extending to models with congestion - Uniqueness

Proposition (Uniqueness)

Extending to models with congestion - Uniqueness

Proposition (Uniqueness)

Assume that g is such that:

$$\forall (m, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (g(i, m) - g(i, \mu))(m_i - \mu_i) \geq 0 \implies m = \mu$$

Extending to models with congestion - Uniqueness

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Assume that the hamiltonian functions can be written as:

$$\forall i \in \mathcal{N}, \forall p \in \mathbb{R}^{d_i}, \forall m \in \mathcal{P}_N, H(i,p,m) = H_c(i,p,m) + f(i,m)$$

with $\forall i \in \mathcal{N}$, $f(i,\cdot)$ *a continuous function satisfying*

$$\forall (m,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (f(i,m) - f(i,\mu))(m_i - \mu_i) \geq 0 \implies m = \mu$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

and $\forall i \in \mathcal{N}$, $H_c(i, \cdot, \cdot)$ *a* C^1 *function with:*
$\forall j \in \mathcal{V}(i), \frac{\partial H_c}{\partial p_j}(i, \cdot, \cdot)$ *a* C^1 *function on* $\mathbb{R}^n \times \mathcal{P}_N$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

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Now, let us define
$A : (q_1, \ldots, q_N, m) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i, m))_{i,j} \in \mathcal{M}_N$
defined by:

$$\alpha_{ij}(q_i, m) = -\frac{\partial H_c}{\partial m_j}(i, q_i, m)$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

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defined by:

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Let us also define, $\forall i \in \mathcal{N}$,
$B^i : (q_i, m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\beta^i_{jk}(q_i, m)\right)_{j,k} \in \mathcal{M}_{N,d_i}$ *defined by:*

$$\beta^i_{jk}(q_i, m) = m_i \frac{\partial^2 H_c}{\partial m_j \partial q_{ik}}(i, q_i, m)$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

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Let us also define, $\forall i \in \mathcal{N}$,
$C^i : (q_i, m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\gamma^i_{jk}(q_i, m)\right)_{j,k} \in \mathcal{M}_{d_i,N}$ *defined by:*

$$\gamma^i_{jk}(q_i, m) = m_i \frac{\partial^2 H_c}{\partial m_k \partial q_{ij}}(i, q_i, m)$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

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Let us finally define, $\forall i \in \mathcal{N}$,
$D^i : (q_i, m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\delta^i_{jk}(q_i, m)\right)_{j,k} \in \mathcal{M}_{d_i}$ *defined by:*

$$\delta^i_{jk}(q_i, m) = m_i \frac{\partial^2 H_c}{\partial q_{ij} \partial q_{ik}}(i, q_i, m)$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

Assume that $\forall (q_1, \dots, q_N, m) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N$

$$\begin{pmatrix} A(q_1,\dots,q_N,m) & B^1(q_1,m) & \cdots & \cdots & \cdots & B^N(q_N,m) \\ C^1(q_1,m) & D^1(q_1,m) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N,m) & 0 & \cdots & \cdots & 0 & D^N(q_N,m) \end{pmatrix} \geq 0$$

Extending to models with congestion - Uniqueness

Proposition (Uniqueness (continued))

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Then, if $(\widehat{u}, \widehat{m})$ *and* $(\tilde{u}, \tilde{m})$ *are two* C^1 *solutions of the* $\mathcal{G}$*-MFG equations, we have* $\widehat{m} = \tilde{m}$ *and* $\widehat{u} = \tilde{u}$*.*

Conclusion

Advantages and limitations of MFGs

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Limitations/Drawbacks

- Only rational/perfect expectations $\rightarrow$ models are not flexible enough sometimes.
- Numerical methods on graphs have not been proposed... maybe **Reinforcement Learning**.

The End

Thank you. Questions?