Fast Marching and Geodesic Methods. Some Applications

Laurent D. COHEN

Directeur de Recherche CNRS
CEREMADE, UMR CNRS 7534 Université Paris-9 Dauphine
Place du Maréchal de Lattre de Tassigny 75016 Paris, France

Cohen@ceremade.dauphine.fr

http://www.ceremade.dauphine.fr/~cohen


Huawei, February 3rd, 2017
Overview

- Minimal Paths, Fast Marching and Front Propagation
- Anisotropic Minimal Paths and Tubular model
- Finding contours as a set of minimal paths
- Application to 2D and 3D tree structures
- Geodesic Density for tree structures
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Paths of minimal energy

Looking for a path along which a feature Potential $P(x,y)$ is minimal

Example: a vessel dark structure
$P =$gray level

Input: Start point $p_1 = (x_1,y_1)$
End point $p_2 = (x_2,y_2)$

Image

Output: Minimal Path
Minimal Paths: Eikonal Equation

\[ E(C) = \int_0^L P(C(s)) ds \]

Potential \( P > 0 \) takes lower values near interesting features: on contours, dark structures, ...

**STEP 1**: search for the surface of minimal action \( U \) of \( p1 \) as the minimal energy integrated along a path between start point \( p1 \) and any point \( p \) in the image

**Startpoint** \( C(0) = p1 \);

\[ U_{p1}(p) = \inf_{C(0)=p1;C(L)=p} E(C) = \inf_{C(0)=p1;C(L)=p} \int_0^L P(C(s)) ds \]

**STEP 2**: Back-propagation from the end point \( p2 \) to the start point \( p1 \):

Simple Gradient Descent along \( U_{p1} \)
Mineral Paths: Eikonal Equation

STEP 1: minimal action $U$ of $pI$ as the minimal energy integrated along a path between start point $pI$ and any point $p$ in the image

Start point $C(0) = pI$;

$$U_{pI}(p) = \inf_{C(0)=pI;C(L)=p} E(C) = \inf_{C(0)=pI;C(L)=p} \int_0^L P(C(s)) ds$$

$$\nabla U_{pI}(x) = P(x) \text{ and } U_{pI}(pI) = 0$$

Example $P=1$, $U$ Euclidean distance to $pI$

in general, $U$ weighted geodesic distance to $pI$
Minimal paths - 2D simple examples

Fermat Principle in Geometric Optics:
Path followed by light minimizes time

where $n > 1$ is refraction index $v = c/n$

Snell-Descartes law

Examples of shortest paths on univalued or bivalued potential
Minimal Paths and Front Propagation

Minimal Action \( U_{p_0}(p) = \inf_{C(0)=p_0; C(L)=p} \int_0^L \tilde{P}(C(s)) \, ds \)

Front Propagation \( \mathcal{L}(t) = \{ p \in \mathbb{R}^2 / U_{p_0}(p) = t \} \)

Evolution of \( t \) level set of \( U \) from \( p_0 \)

\[
\frac{\partial \mathcal{L}(\sigma, t)}{\partial t} = \frac{1}{P(\mathcal{L}(\sigma, t))} \mathbf{n}(\sigma, t)
\]

\( n \) normal vector to a level set of \( U \) is in the direction of the Gradient of \( U \), implies Eikonal Equation:

\[
\| \nabla U_{p_0}(x) \| = P(x) \text{ and } U_{p_0}(p_0) = 0
\]
FAST MARCHING in 2D:
very efficient algorithm $O(N\log N)$ for Eikonal Equation

Introduced by Sethian / Tsitsiklis
Numerical approximation of $U(x_{ij})$ as the solution to the discretized problem with upwind finite difference scheme

$$\|\nabla U\| = \tilde{P}$$

$$\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 = \tilde{P}^2$$

$$\max \left( u - U(x_{i-1,j}), u - U(x_{i+1,j}), 0 \right)^2$$
$$+ \max \left( u - U(x_{i,j-1}), u - U(x_{i,j+1}), 0 \right)^2 = h^2 \tilde{P}(x_{i,j})^2$$

This 2nd order equation induces that:
action $U$ at $\{i,j\}$ depends only of the neighbors that have lower actions.
Fast marching introduces order in the selection of the grid points for solving this numerical scheme.
Starting from the initial point $p_1$ with $U = 0$,
the action computed at each point visited can only grow.

Level sets of $U$ can be seen as a Front propagation outwards.
Fast Marching Algorithm (Sethian)

- **Start**: only $p_0$ is trial with $U=0$.
- **Loop**: $p$ trial point with minimum $U$ becomes alive. neighbors of $p$ become trial and are updated.
Fast Marching Algorithm

Initialization

J. A. Sethian

A fast marching level set method for monotonically advancing fronts.

Fast Marching Algorithm

Itération #1

- Find point $x_{\text{min}}$ (Trial point with smallest value of $\mathcal{U}$).
- $x_{\text{min}}$ becomes Alive.
- For each of 4 neighbors $x$ of point $x_{\text{min}}$:
  - If $x$ is not Alive, estimate $\mathcal{U}(x)$ with upwind scheme.
  - $x$ becomes Trial.

J. A. Sethian

A fast marching level set method for monotonically advancing fronts.
Fast Marching Algorithm

Itération #2
- Find point $x_{\text{min}}$ (Trial point with smallest value of $U$).
- $x_{\text{min}}$ becomes Alive.
- For each of 4 neighbors $x$ of point $x_{\text{min}}$:
  - If $x$ is not Alive,
    - Estimate $U(x)$ with upwind scheme.
    - $x$ becomes Trial.

J. A. Sethian
A fast marching level set method for monotonically advancing fronts.
Fast Marching Algorithm

\textbf{Itération \#k}

- Find point $x_{\text{min}}$ (\textit{Trial} point with smallest value of $U$).
- $x_{\text{min}}$ becomes \textit{Alive}.
- For each of 4 neighbors $x$ of point $x_{\text{min}}$:
  - If $x$ is not \textit{Alive},
    - Estimate $U(x)$ with upwind scheme.
  - $x$ becomes \textit{Trial}.

---

J. A. Sethian

\textit{A fast marching level set method for monotonically advancing fronts.}

\textit{P.N.A.S., 93:1591-1595, 1996.}
Minimal Path between p1 and p2

image \( I \)

© C. Bouzigues

potential \( \mathcal{P} : \Omega \to \mathbb{R}^+ \)
Minimal Path between $p_1$ and $p_2$

Step #1

\[
\begin{cases}
\| \nabla U_1(x) \| = \tilde{P}(x) \text{ pour } x \in \Omega \\
U_1(p_1) = 0
\end{cases}
\]
Minimal Path between $p_1$ and $p_2$

Step #1: $U$ obtained by the FAST MARCHING ALGORITHM

$$\begin{cases} 
||\nabla U_1(x)|| = \tilde{P}(x) \quad \text{pour} \ x \in \Omega \\
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L. D. Cohen, R. Kimmel
*Global minimum for active contour models: a minimal path approach.*
Step #1: $U$ obtained by the FAST MARCHING ALGORITHM

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Minimal Path between $p_1$ and $p_2$
Minimal Path between p1 and p2

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Step #2

gradient descent on \(U_1\) for extraction of minimal path \(C_{p_1,p_2}\)

\[
\begin{align*}
\frac{\partial C_{p_1,p_2}(s)}{\partial s} &= -\nabla U_1(C_{p_1,p_2}(s)) \\
C_{p_1,p_2}(0) &= p_2
\end{align*}
\]
Minimal Path between $p_1$ and $p_2$

L. D. Cohen, R. Kimmel

Global minimum for active contour models: a minimal path approach.

Minimal Path between p1 and p2

\[ C_{p_1, p_2} = \min_{\gamma \in A_{p_1, p_2}} \int \dot{\gamma}(s) \, ds \]

Is obtained by solving ODE:

\[
\begin{cases}
\frac{\partial C_{p_1, p_2}(s)}{\partial s} = -\nabla U_1(C_{p_1, p_2}(s)) \\
C_{p_1, p_2}(0) = p_2
\end{cases}
\]

\( \Rightarrow \) simple gradient descent on \( U_1 \) from \( p_2 \) to \( p_1 \)

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Global minimum for active contour models: a minimal path approach.

Minimal Path between p1 and p2

Step #1

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Minimal Path between \( p_1 \) and \( p_2 \)

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\end{align*}
\]
Minimal paths for 2D segmentation

\[ P(x) = 1 \implies \text{droite (plus court chemin euclidien)} \]
Minimal paths for 2D segmentation

\[ P(x) = w + (I(x) - I(x_0))^2 \iff \text{chemin d'intensité homogène} \]
Simultaneous propagation from both ends

Reference:
T. Deschamps and L. D. Cohen
Minimal paths in 3D images and application to virtual endoscopy.
Proceedings ECCV'00, Dublin, Ireland, 2000.
Simultaneous propagation of two fronts until a shock occurs.

Reference:
T. Deschamps and L. D. Cohen
Minimal paths in 3D images and application to virtual endoscopy.
**Link with Dynamic Programming**

**- Metrication error -**

**Fig. 22:** An $L^1$ norm cause the shortest path to suffer from errors of up to 41%. In this case both $P_1$ and $P_2$ are optimal, and will stay optimal no matter how much we refine the (4-neighborhood) grid.
Fig. 23: Illustration of metrization error for computation of the distance map to a single point, showing level sets of the distance. On the left: a graph search-like discrete distance computation gives squares; on the right: the distance is obtained by our approach, giving circles.
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3D Minimal Path for tubular shapes in 2D Centerline+width
2D in space, 1D for radius of vessel (Li, Yezzi 2007)

Figure 1. A tubular surface is presented as the envelope of a family of spheres with continuously changing center points and radii.
3D Minimal Path for tubular shapes in 2D

2D in space, 1D for radius of vessel

Fig. 2. Vessel segmentation for an angiogram 2D projection image based on the proposed method
Typical Retina Image
Two pairs of user given points
Extraction by 2D+radius minimal path model
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Anisotropic Energy

Considers the local orientations of the structures

\[ E(C) = \int_0^L P(C(s), C'(s)) ds \]

Describes an infinitesimal distance along an oriented pathway \( C \), relative to a metric \( H \)

\[ P(C(s), C'(s)) = \sqrt{C'(s)^T H(C(s)) C'(s)} \]

Anisotropic Energy: Eikonal Equation

\[ E(C) = \int_0^L \sqrt{C'(s)^T H(C(s))C'(s)} \, ds \]

Start point \( C(0) = p_1 \); \[ U_{p_1}(p) = \inf_{C(0)=p_1:C(L)=p} E(C) \]

\[ \| \nabla U_{p_1}(p) \|_{H(p)^{-1}} = \sqrt{\nabla U_{p_1}^T H^{-1} \nabla U_{p_1}} = 1 \]

and \( U_{p_1}(p_1) = 0 \)

Anisotropic Energy: Gradient descent

\[ E(C) = \int_0^L \sqrt{C'(s)^T H(C(s)) C'(s)} \, ds \]

Start point \( C(0) = p_1 \);

\[ U_{p_1}(p) = \inf_{C(0)=p_1; C(L)=p} E(C) \]

\[ C'(s) = -H^{-1}(C(s)) \nabla U_{p_1}(C(s)) \]

and \( U_{p_1}(p_1) = 0 \)

Anisotropic Energy: includes Isotropic case

\[ E(C) = \int_0^L \sqrt{C'(s)^T H(C(s)) C''(s)} \, ds \]

Start point \( C(0) = p_1 \);

\[ H(p) = P^2(p) I_d \]

\[ \| \nabla U_{p_1}(p) \| = P \]

\[ C'(t) = -\nabla U_{p_1}(C(t)) \]

Anisotropy and Geodesics

Tensor eigen-decomposition:

\[ H(x) = \lambda_1(x)e_1(x)e_1(x)^T + \lambda_2(x)e_2(x)e_2(x)^T \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2, \]

\[ \{ \eta \mid \eta^*H(x)\eta \leq 1 \} \]

\[ \lambda_2(x)^{-\frac{1}{2}} \quad e_2(x) \]

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Anisotropy and Geodesics

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\[ \{ \eta \mid \eta^*H(x)\eta \leq 1 \} \]

\[ \lambda_2(x)^{-\frac{1}{2}} \]

\[ \lambda_1(x)^{-\frac{1}{2}} \]

Geodesics tend to follow \( e_1(x) \).
Anisotropy and Geodesics

FIG. 2.14: Given an elliptic metric $M = w_1^2 e_r e_r^T + w_2^2 e_\theta e_\theta^T$ with standard polar notations, influence of anisotropy ratio $\frac{w_2}{w_1}$ is shown.
Orientation-Dependent Energy
(with Benmansour, CVPR'09, IJCV'10)

\[ E(C) = \int_{0}^{L} P(C(s), C'(s)) ds \]

Considers the local orientations of the structures
Examples of 3D Minimal Paths for tubular shapes in 2D

2D in space, 1D for radius of vessel
Examples of 3D Minimal Paths for tubular shapes in 2D

2D in space, 1D for radius of vessel
Examples of 4D Minimal Paths for tubular shapes in 3D
Examples of 4D Minimal Paths for tubular shapes in 3D

3D in space , 1D for radius of vessel
Perceptual Grouping using Minimal Paths

The potential is an incomplete ellipse and 7 points are given (keypoints were found using a Furthest point strategy).

Reference:
L. D. Cohen
Multiple Contour Finding and Perceptual Grouping using Minimal Paths.
Perceptual Grouping using a set of Minimal Paths
Perceptual Grouping using Minimal Paths

Reference:
L. D. Cohen
Multiple Contour Finding and Perceptual Grouping using Minimal Paths.
Perceptual Grouping using Minimal Paths
Using the orientation with anisotropic geodesics

Application Endoscopie Virtuelle (collaboration Philips Recherche)
Vue Endoscopique d’un arbre vasculaire
Curvature Penalized Minimal Path Method with A Finsler Metric
with Da Chen and JM Mirebeau, 2015-2016

- The metric may depend on the orientation
- Orientation-lifted metric: the curve length of Euler elastica can be exactly computed by this metric
Curvature Penalized Minimal Path Method with A Finsler Metric
Curvature Penalized Minimal Path Method with a Finsler Metric
Curvature Penalized Minimal Path Method with A Finsler Metric
Fig. 8 Geodesics extraction results using the proposed Finsler metric. Red and green dots are the initial and end positions respectively. Arrows indicate the corresponding tangents.
Curvature Penalized Minimal Path Method with A Finsler Metric

Fig. 9 Comparative closed contour detection results. **Column 1**: edge saliency map. **Columns 2-5**: results from the IR metric, the AR metric, the IOLR metric and the proposed Finsler metric. In Column 5, arrows indicate the tangents for the corresponding physical positions denoted by dots.
Curvature Penalized Minimal Path Method with A Finsler Metric

Fig. 10 Closed contour detection results using only two given physical positions.
Examples of Remeshing

Original mesh  Uniform  Curvature adapted
Isotropic vs. Anisotropic Meshing
Anisotropic Meshing

![Anisotropic Meshing Diagram](image)

farthest point strategy
Anisotropic Meshing

farthest point strategy
Examples of Anisotropic Meshing
controls density and orientation of triangles
Geodesic methods for shape recognition
Based on distribution (histogram) of geodesic distances
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Finding a closed contour by growing minimal paths and adding keypoints

Finding a closed contour by growing minimal paths and adding keypoints

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Adding keypoints: Stopping criterion

The propagation must be stopped as soon as the domain visited by the fronts has the same topology as a ring.
**Finding a closed contour by growing minimal paths and adding keypoints**

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Laurent D. COHEN, Huawei 2017
Finding a closed contour by growing minimal paths
Keypoints and 3D Minimal Paths for tubular shapes in 2D
(with Li and Yezzi, MICCAI’09)

Fig. 1. The entire multi-branch structure extraction is reduced to finding structures between all adjacent key point pairs. The 4D path length $D$ between each key point pair is equal to $d_{step}$. For easier visualization, the same concept is illustrated here using circles instead of spheres.
Keypoints and 3D Minimal Paths for tubular shapes in 2D

2D in space, 1D for radius of vessel
Keypoints and 3D Minimal Paths for tubular shapes in 2D
Keypoints and 3D Minimal Paths for tubular shapes in 2D

Fig. 3. Segmentation results via the proposed method on another 2D projection angiogram image. Panels from left to right show the initial point and the detected iterative key points and the detected vessel surfaces.
Automatic Keypoint Growing with Mask
(with Chen Da)
Automatic Keypoint Method
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Geodesic Density
Geodesic Density
Geodesic Density
Geodesic Density: Real example
Geodesic Density: adaptive voting

Adaptive voting: 1000 end points
Geodesic Density: adaptive voting

Adaptive voting : 1000 end points
Conclusion

- Minimally interactive tools for vessels and vascular tree segmentation (tubular branching structures)
- User provides only one initial point and sometimes second end point or stopping parameter
- Fast and efficient propagation algorithm
- Models may include orientation and scale of vessels
- Voting approach as a powerful tool to find the structure, which can be completed with other approach.
Thank you!

Cohen@ceremade.dauphine.fr

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